THE MONOIDAL INTERVAL FOR THE MONOID GENERATED BY TWO CONSTANTS

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Abstract. In 1941, Post presented the complete description of the countably many clones on a 2-element set. The structure of the lattice of clones on sets of finitely many (but more than 2) elements is more complex; in fact, the lattice is of cardinality $2^{\aleph_0}$. One approach is to study the monoidal intervals: the set of clones whose unary operations form a given monoid. In this article, we study the monoidal interval for the monoid generated by two constants on sets of k elements for k finite. This interval contains the clones of term operations of the bounded lattices of k elements.

1. **Preliminaries**

Let A be a finite set and n a positive integer. An *n-ary operation* on A is a function $f : A^n \to A$. The set of all n-ary operations on A is denoted by $\mathcal{O}_A^{(n)}$, and $\mathcal{O}_A := \bigcup_{0<n<\omega} \mathcal{O}_A^{(n)}$. For $F \subseteq \mathcal{O}_A$, set $F^{(n)} := F \cap \mathcal{O}_A^{(n)}$. For $1 \leq i \leq n$, the n-ary *i-th projection* is defined as $e_i^{(n)}(x_1, \ldots, x_n) = x_i$ for all $x_1, \ldots, x_n$. We write e for the identity operation. For $a \in A$, the n-ary *constant operation a* is defined as $c_a^{(n)}(x_1, \ldots, x_n) = a$ for all $x_1, \ldots, x_n$. We write simply c_a for the unary constant operations $c_a^{(1)}$.

For $f \in \mathcal{O}^{(n)}$, and $g_1, \ldots, g_n \in \mathcal{O}^{(m)}$, we define their composition to be the m-ary operation $f[g_1, \ldots, g_n]$ defined by

$$f[g_1, \ldots, g_n](x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$$

A *clone* on A is a subset F of $\mathcal{O}_A$ that contains all projections and is closed under composition. It is well known and easy to prove that the intersection of an arbitrary set of clones on A is a clone on A. Thus for $F \subseteq \mathcal{O}_A$, there exists the least clone containing F, called the clone *generated* by F and denoted by $\langle F \rangle$. Equivalently, $\langle F \rangle$ is the set of term operations of the algebra $\langle A; F \rangle$. The clones on A, ordered by inclusion, form a complete lattice, $\mathcal{L}_A$.

Let h be a positive integer. An *h-ary relation* ρ is a subset of A^h. When dealing with a fixed $\rho \in A^2$, we write $a \to b$ for $(a, b) \in \rho$. The relations may then be drawn as directed graphs. For example for $A = \{0, 1, 2\}$, the relation $\{(0,0), (0,1), (1,0), (1,2)\}$ may be represented as in Figure 1

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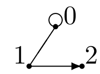

FIGURE 1. Example of a relation

Let $f \in \mathcal{O}^{(n)}$, and let ρ be an h-ary relation on A. The operation f *preserves* ρ if for all $(a_{1,i}, a_{2,i}, \ldots, a_{h,i}) \in \rho$ $(i = 1, \ldots, n)$,

$$(f(a_{1,1}, a_{1,2}, \ldots, a_{1,n}), f(a_{2,1}, a_{2,2}, \ldots, a_{2,n}), \ldots, f(a_{h,1}, a_{h,2}, \ldots, a_{h,n})) \in \rho$$

The set of operations on A preserving ρ is a clone denoted by $\operatorname{Pol}\rho$.

Consider a transformation monoid M of unary operations on A; i.e. M contains the identity self-map e and is closed under the usual composition. Denote by $\operatorname{Int}(M)$ the set of clones C on A such that $C^{(1)} = M$. It is well known that $\operatorname{Int}(M)$ is an interval in the lattice of clones on A, called the *monoidal interval* of M. The smallest clone in $\operatorname{Int}(M)$ is $\langle M \rangle$. The largest clone in $\operatorname{Int}(M)$ is the *stabilizer* of M:

$$\begin{aligned}\operatorname{Sta}(M) &= \{f \in \mathcal{O}_A^{(n)} \mid n > 0 \text{ and } f[m_1, \ldots, m_n] \in M \text{ for all } m_1, \ldots m_n \in M\} \\ &= \operatorname{Pol}\{(m(a_1), \ldots, m(a_k)) \mid m \in M\}\end{aligned}$$

for $A = \{a_1, \ldots, a_k\}$ finite (see [1]). If $\operatorname{Int}(M)$ contains just one clone, i.e. $\operatorname{Int}(M) = \{\langle M \rangle\}$, then M is said to be *collapsing*.

2. **Motivation**

In 1995, Krokhin [2] showed that the monoidal interval $\operatorname{Int}\langle c_0, c_1, \ldots c_{k-2}\rangle$ on $A = \{0, 1, \ldots, k-1\}$ is collapsing for $k > 3$. For $k = 3$, he showed that the monoidal interval $\operatorname{Int}\langle c_0, c_1\rangle$ has 4 clones. These correspond to the 3 types of algebras ennumerated by Berman, with free-spectra begining with 2,3 [3]. Its diamond shaped Hasse diagram is in Figure 2. Here min and max are defined with respect to the chain $0 < 2 < 1$.

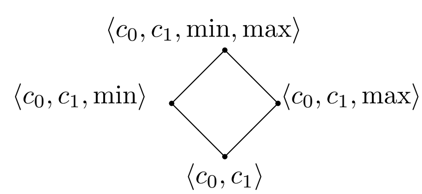

FIGURE 2. The interval $\operatorname{Int}\langle c_0, c_1\rangle$ on a 3-element universe

We propose to generalize the interval of Figure 2 in a different way, by considering the interval $\operatorname{Int}\langle c_0, c_1\rangle$ for universes of at least 2 elements.

On $\{0, 1\}$, Post [4] showed that the interval $\operatorname{Int}\langle c_0, c_1\rangle$ is the one in Figure 3 where $\wedge$ and $\vee$ are min and max with respect to the chain $0 < 1$, or equivalently, the conjunction and disjunction from symbolic logic.

We show in Section 5, that for any finite bounded distributive lattice $\langle A; \vee, \wedge, 0, 1\rangle$, the interval $[\langle c_0, c_1\rangle, \langle c_0, c_1, \vee, \wedge\rangle]$ in the lattice of clones on A has the same Hasse diagram as in Figure 3. Note that for A finite and $|A| > 3$, it is well

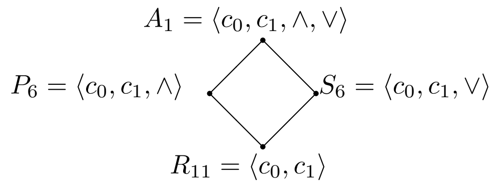

FIGURE 3. The interval $\text{Int}\langle c_0, c_1\rangle$ on a 2-element universe

known that $|\,\text{Int}\langle c_0, c_1\rangle| = 2^{\aleph_0}$ [2]. Thus the diamond shaped intervals mentioned above cannot be the whole interval $\text{Int}\langle c_0, c_1\rangle$, but they do appear in the bottom of it.

In [5], it was shown that, for $A = \{0, 1, 2\}$, the clone

$$\text{Pol}\{(0,0),(1,1),(0,1),(1,2),(2,0)\} = \langle c_0, c_1\rangle$$

which is the smallest clone in the interval. In Section 3, we generalize this result to universes of k elements for $k \geq 3$. Note that this is a non-reflexive strongly C-rigid relation [6] (i.e. a relation preserved only by constants and permutations).

Combining results from [2] and [5] for $A = \{0, 1, 2\}$, it is easy to show that

$$\text{Pol}\{(0,0),(1,1),(1,2),(2,0)\} = \langle c_0, c_1, \min, \max\rangle$$

which is the largest clone in the interval. In Section 4, we exhibit a relation that is preserved by the clone at the top of each min, max diamond for larger universes. This theorem even works for sets of 2 elements; it is the well-known result that $\text{Pol}(\leq) = \langle c_0, c_1, \wedge, \vee\rangle$ (see for example [1]).

3. **A relation preserved by the two constants**

In this section we exhibit a relation for the smallest clone in the monoidal interval $\text{Int}\langle c_0, c_1\rangle$. This result is a generalization of a theorem in [5], which dealt with the 3-element case and stated that $\text{Pol}\{(0,0),(1,1),(0,1),(1,2),(2,0)\} = \langle c_0, c_1\rangle$.

This result ties into research by Länger and Pöschel. A relation ρ is *strongly C-rigid* if every operation on A preserving ρ is a projection or a constant function. Länger and Pöschel [6] presented many reflexive strongly C-rigid relations. The relation in Theorem 1 is a strongly C-rigid relation that is not reflexive.

Theorem 1. *Let $k \geq 3$ and $A = \{0, 1, \ldots, k-1\}$. Let*

$$\rho = \{(0,0),(1,1),(0,1),(1,2),(2,3),\ldots,(k-1,0)\}$$

Then $\text{Pol}\,\rho = \langle c_0, c_1\rangle$.

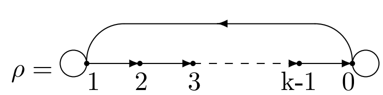

FIGURE 4

As an oriented graph, ρ consists of a k-cycle and two loops (Figure 4).

Before proving the theorem, let us state a pair of definition and prove some lemmas. We define two unary operations on A: $x^{\rightarrow}$ defined by $0^{\rightarrow} = 0$, $1^{\rightarrow} = 1$, $j^{\rightarrow} = j + 1$ if $1 < j < k - 1$ and $(k-1)^{\rightarrow} = 0$, and $x^{\leftarrow}$ defined by $0^{\leftarrow} = 0$, $1^{\leftarrow} = 1$ and $j^{\leftarrow} = j - 1$ if $1 < j \leq k - 1$. We write $a^{i\leftarrow}$ and $a^{i\rightarrow}$ for the composition i times of the arrow operations. The following proposition follows from the definitions.

Proposition 2. (A) $a^{\leftarrow} \to a \to a^{\rightarrow}$ *for all* $a \in A$.
(B) $a^{i\leftarrow} \leftarrow a^{(i+1)\leftarrow}$ *and* $a^{i\rightarrow} \to a^{(i+1)\rightarrow}$ *for all* $a \in A$ *and* $i \in \{1, 2, \ldots\}$.
(C) $a^{(k-2)\leftarrow}, a^{(k-2)\rightarrow} \in \{0, 1\}$ *for all* $a \in A$.

Lemma 3. *Let* $f \in \operatorname{Pol}\rho$ *be an* n*-ary function and let* $x_1, \ldots, x_n \in \{0,1\}$. *Then* $f(x_1, \ldots, x_n) \in \{0, 1\}$.

Proof. For $m = 1, \ldots, n$, $x_m \leftrightarrow x_m$. Thus $f(x_1, \ldots, x_n) \leftrightarrow f(x_1, \ldots, x_n)$. Therefore $f(x_1, \ldots, x_n) \in \{0, 1\}$. □

Lemma 4. *The unary functions in* $\operatorname{Pol}\rho$ *are exactly* c_0, c_1 *and* e.

Proof. Note that $c_0, c_1, e \in \operatorname{Pol}\rho$.

Let $f \in (\operatorname{Pol}\rho)^{(1)}$. By lemma 3, $f(0), f(1) \in \{0, 1\}$.

Case 1: If $f(0) = 1$, we have $1 = f(0) \to f(1)$, which along with Lemma 3, implies that $f(1) = 1$. Now $1 = f(1) \to f(2) \to \ldots \to f(k-1) \to f(0) = 1$; a $(k-1)$-cycle. This implies that $f(2) = \ldots = f(k-1) = 1$. Therefore $f = c_1$.

Case 2: If $f(1) = 0$, we obtain that $f = c_0$ by a similar reasoning.

Case 3: If $f(0) = 0$ and $f(1) = 1$, we have $1 = f(1) \to f(2) \to \ldots \to f(k-1) \to f(0) = 0$. This implies that $f(2) = 2, \ldots, f(k-1) = k-1$. Therefore $f = e$. □

Proof of Theorem 1. For every $f \in (\operatorname{Pol}\rho)^{(n)}$, we consider the corresponding Boolean function $f|_{\{0,1\}} : \{0,1\}^n \to \{0,1\}$. This is possible because of Lemma 3. Note that the constants for sets of k elements correspond to the Boolean constants. Now define $(\operatorname{Pol}\rho)|_{\{0,1\}} := \{f|_{\{0,1\}} \mid f \in \operatorname{Pol}\rho\}$. Clearly, $(\operatorname{Pol}\rho)|_{\{0,1\}}$ is a clone on $\{0, 1\}$.

Claim 1. $(\operatorname{Pol}\rho)|_{\{0,1\}} = \langle c_0, c_1 \rangle$.

Proof. We use Post's classification [4]. Since $c_0, c_1 \in (\operatorname{Pol}\rho)|_{\{0,1\}}$, clearly, $R_{11} \subseteq (\operatorname{Pol}\rho)|_{\{0,1\}}$.

By Lemma 4, the unary functions in $(\operatorname{Pol}\rho)|_{\{0,1\}}$ are exactly c_0, c_1 and e. Thus $\neg \notin (\operatorname{Pol}\rho)|_{\{0,1\}}$, which implies that $R_4 \nsubseteq (\operatorname{Pol}\rho)|_{\{0,1\}}$

Suppose for the sake of contradiction that $\wedge \in (\operatorname{Pol}\rho)|_{\{0,1\}}$. Then there must be some $f \in \operatorname{Pol}\rho$ such that $f|_{\{0,1\}} = \wedge$. We thus have

$$1 = f(1,1) \to f(1,2) \to f(1,3) \to \ldots \to f(1, k-1) \to f(1,0) = 0$$

which implies that $f(1, a) = a$ for all $a \in A$. It follows that $0 = f(0, 1) \to f(1, 2) = 2$, which is impossible. Therefore $\wedge \notin (\operatorname{Pol}\rho)|_{\{0,1\}}$, which implies that $P_2 \nsubseteq (\operatorname{Pol}\rho)|_{\{0,1\}}$. Similarly $\vee \notin (\operatorname{Pol}\rho)|_{\{0,1\}}$, which implies that $S_2 \nsubseteq (\operatorname{Pol}\rho)|_{\{0,1\}}$. Therefore $(\operatorname{Pol}\rho)|_{\{0,1\}} = R_{11} = \langle c_0, c_1 \rangle$ as required. □

Claim 2. $\operatorname{Pol}\rho \subseteq \langle c_0, c_1 \rangle$.

Proof. Let $f \in (\operatorname{Pol}\rho)^{(n)}$. By the previous Claim, $f|_{\{0,1\}} \in \langle c_0, c_1 \rangle$, thus $f|_{\{0,1\}}$ must be c_0^n or c_1^n or e_m^n for some $m \in \{1, \ldots, n\}$.

Case 1: $f|_{\{0,1\}} = c_0^n$. We will show by induction on i, that $f|_{\{0,\ldots,i\}} = c_0^n$ for all $i \in \{1, \ldots, k-1\}$. The statement is true for $i = 1$.

Now suppose that $f|_{\{0,\ldots,j\}} = c_0^n$ for a certain j with $1 \le j < k-2$. We must prove that $f|_{\{0,\ldots,j+1\}} = c_0^n$. Let $x_1, \ldots, x_n \in \{0, \ldots, j+1\}$. Note that $x_1^{\leftarrow}, \ldots, x_n^{\leftarrow} \in \{0, \ldots, j\}$. By Proposition 2 and the induction hypothesis, $0 = c_0^n(x_1^{\leftarrow}, \ldots, x_n^{\leftarrow}) = f(x_1^{\leftarrow}, \ldots, x_n^{\leftarrow}) \to f(x_1, \ldots, x_n) \to f(x_1^{\rightarrow}, \ldots, x_n^{\rightarrow}) \to \ldots \to f(x_1^{(k-2)\rightarrow}, \ldots, x_n^{(k-2)\rightarrow}) = c_0^n(x_1^{(k-2)\rightarrow}, \ldots, x_n^{(k-2)\rightarrow}) = 0$; a $(k-1)$-cycle, so they are all 0. In particular, $f(x_1, \ldots, x_n) = 0$, which implies that $f|_{\{0,\ldots,j+1\}} = c_0^n$ as required. Therefore, $f = c_0^n \in \langle c_0, c_1 \rangle$, by induction.

CASE 2: $f|_{\{0,1\}} = c_1^n$. Then $f = c_1^n \in \langle c_0, c_1 \rangle$ in a similar way.

CASE 3: $f|_{\{0,1\}} = e_m^n$ on $\{0,1\}$ for some $m \in \{1, \ldots, n\}$. Without loss of generality, $f|_{\{0,1\}} = e_1^n$.

We will show by induction on i, that $f|_{\{0,\ldots,i\}} = e_1^n$ for all $i \in \{1, \ldots, k-1\}$. The statement is true for $i = 1$.

Now suppose that $f|_{\{0,\ldots,j\}} = e_1^n$ for a certain j with $1 \le j < k-2$. We must prove that $f|_{\{0,\ldots,j+1\}} = e_1^n$. Let $x_1, \ldots, x_n \in \{0, \ldots, j+1\}$. Note that $x_1^{\leftarrow}, \ldots, x_n^{\leftarrow} \in \{0, \ldots, j\}$. Therefore, $f(x_1, \ldots, x_n) \leftarrow f(x_1^{\leftarrow}, \ldots, x_n^{\leftarrow}) = x_1^{\leftarrow}$. The only way for $f(x_1, \ldots, x_n) \neq x_1$ would be if $x_1 = 0$ and $f(x_1, \ldots, x_n) = 1$, or if $x_1 = 1$ and $f(x_1, \ldots, x_n) = 2$, or if $x_1 = 2$ and $f(x_1, \ldots, x_n) = 1$. By Proposition 2,

$$(1) \quad f(x_1, \ldots, x_n) \to f(x_1^{\rightarrow}, \ldots, x_n^{\rightarrow}) \to \ldots \to f(x_1^{(k-2)\rightarrow}, \ldots, x_n^{(k-2)\rightarrow}) = x_1^{(k-2)\rightarrow}$$

which is a chain of length $k-2$. If $x_1 = 0$ and $f(x_1, \ldots, x_n) = 1$, then (1) becomes $1 \to \ldots \to 0$, which is impossible. If $x_1 = 1$ and $f(x_1, \ldots, x_n) = 2$, then (1) becomes $2 \to \ldots \to 1$, which is impossible. If $x_1 = 2$ and $f(x_1, \ldots, x_n) = 1$, then (1) becomes $1 \to \ldots \to 2^{(k-2)\rightarrow} = 0$, which is impossible. Therefore $f(x_1, \ldots, x_n) = x_1$ as required. □

Lemma 4 implies that $\langle c_0, c_1 \rangle \subseteq \operatorname{Pol} \rho$. Using this and Claim 2, we conclude that $\operatorname{Pol} \rho = \langle c_0, c_1 \rangle$. □

4. **Another clone in the interval**

Theorem 1 gave a relation for the bottom of the interval. The following theorem describes a relation for the top of the diamond. Note that for 3-element sets, the relation is $\{(0,0), (1,1), (1,2), (2,0)\}$ as expected. Although the theorem is not proved for 2-element sets in the same way, we will show that the relation is indeed $\{(0,0), (1,1), (1,0)\}$.

Theorem 5. *Let $k \ge 3$ and $A = \{0, 1, \ldots, k-1\}$. Let*

$$\sigma = \{(0,0), (1,1), (1,2), (2,3), \ldots, (k-1, 0)\}$$

Then $\operatorname{Pol} \sigma = \langle c_0, c_1, \min, \max \rangle$ where $\min$ and $\max$ are defined according to the chain $0 < k-1 < k-2 < \ldots < 2 < 1$.

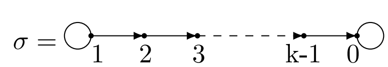

FIGURE 5

As an oriented graph, σ consists of a chain of length $k-1$ and two loops at its end points (Figure 5).

We will use the operators $x^{\rightarrow}$ and $x^{\leftarrow}$ from Theorem 1. Note that Proposition 2 holds for σ.

Lemma 6 (Post [4], see also [1]). *If $k = 2$, then $\operatorname{Pol}\sigma = \langle c_0, c_1, \min, \max\rangle = A_1$, the maximal clone of monotone Boolean functions*

Lemma 7. *Let $f \in \operatorname{Pol}\sigma$ be an n-ary function and let $x_1, \dots, x_n \in \{0, 1\}$. Then $f(x_1, \dots, x_n) \in \{0, 1\}$.*

Proof. Identical to the proof of Lemma 3. □

Lemma 8. *The unary functions in $\operatorname{Pol}\sigma$ are exactly c_0, c_1 and e.*

Proof. Almost identical to the proof of Lemma 4. □

Proof of Theorem 5. The Theorem is true for $k = 2$ by Lemma 6. From now on, we assume that $k \geq 3$.

Claim 1. *c_0, c_1, $\min$ and $\max$ are in $\operatorname{Pol}\sigma$.*

Proof. By Lemma 8, we know that $c_0, c_1 \in \operatorname{Pol}\sigma$.

Let $a_1, a_2, b_1, b_2 \in A$ such that $a_1 \rightarrow a_2$ and $b_1 \rightarrow b_2$. If $a_1 = a_2 = 0$, then $\min(a_1, b_1) = \min(0, b_1) = 0 \rightarrow 0 = \min(0, b_2) = \min(a_2, b_2)$. If $a_1 = a_2 = 1$, then $\min(a_1, b_1) = \min(1, b_1) = b_1 \rightarrow b_2 = \min(1, b_2) = \min(a_2, b_2)$. Similarly, if $b_1 = b_2$, we have that $\min(a_1, b_1) \rightarrow \min(a_2, b_2)$. The last case is if $a_2 = a_1 + 1$, $b_2 = b_1 + 1$ and $a_1, b_1 \neq 0$. Here $\min(a_1, b_1) \rightarrow \min(a_1 + 1, b_1 + 1) = \min(a_2, b_2)$. In all cases $\min(a_1, b_1) \rightarrow \min(a_2, b_2)$.

Similarly, we find that $\max(a_1, b_1) \rightarrow \max(a_2, b_2)$. □

For every $f \in (\operatorname{Pol}\sigma)^{(n)}$, we consider the corresponding Boolean function $f|_{\{0,1\}} : \{0, 1\}^n \rightarrow \{0, 1\}$. This is possible because of Lemma 7. Note that $\min|_{\{0,1\}} = \wedge$, $\max|_{\{0,1\}} = \vee$ and that the constants become the corresponding Boolean constants. Now define $(\operatorname{Pol}\sigma)|_{\{0,1\}} := \{f|_{\{0,1\}} \mid f \in \operatorname{Pol}\sigma\}$. Clearly, $(\operatorname{Pol}\sigma)|_{\{0,1\}}$ is a clone on $\{0, 1\}$.

Claim 2. $(\operatorname{Pol}\sigma)|_{\{0,1\}} = \langle c_0, c_1, \wedge, \vee\rangle$ *the maximal clone of monotone Boolean functions.*

Proof. Since c_0, c_1, $\wedge$ and $\vee$ are in $(\operatorname{Pol}\sigma)|_{\{0,1\}}$, clearly, $A_1 \subseteq (\operatorname{Pol}\sigma)|_{\{0,1\}}$. By [4], $(\operatorname{Pol}\sigma)|_{\{0,1\}}$ must be A_1 or C_1, the clone of all Boolean functions.

By Lemma 8, the unary functions in $(\operatorname{Pol}\sigma)|_{\{0,1\}}$ are exactly c_0, c_1 and e, thus $\neg \notin (\operatorname{Pol}\sigma)|_{\{0,1\}}$. Therefore $(\operatorname{Pol}\sigma)|_{\{0,1\}} = A_1 = \langle c_0, c_1, \wedge, \vee\rangle$. □

Claim 3. $\operatorname{Pol}\sigma \subseteq \langle c_0, c_1, \min, \max\rangle$.

Proof. Let $f \in (\operatorname{Pol}\sigma)^{(n)}$. By Claim 2, $f|_{\{0,1\}} \in \langle c_0, c_1, \wedge, \vee\rangle$. Therefore, $f|_{\{0,1\}}$ can be written as a term using the functions c_0, c_1, $\wedge$ and $\vee$. Define a new function $g : A^n \rightarrow A$ from the term for $f|_{\{0,1\}}$ by replacing all occurrences of the constants by the corresponding constants on A, and by replacing all occurrences of $\wedge$ and $\vee$ by min and max respectively. Clearly, $g \in \operatorname{Pol}\sigma$. We will prove that $g = f$. In fact, we will prove by induction on i, that for all $i = 1, \dots, k-1$ and for $x_1, \dots, x_n \in \{0, 1, \dots, i\}$, we have $f(x_1, \dots, x_n) \in \{0, 1, \dots, i\}$ and $g(x_1, \dots, x_n) = f(x_1, \dots, x_n)$, For $i = 1$, we know by Lemma 7 that $f(x_1, \dots, x_n) \in \{0, 1\}$ for all $x_1, \dots, x_n \in \{0, 1\}$, and it is trivially true that $g|_{\{0,1\}} = f|_{\{0,1\}}$.

Suppose that for some $j \in \{0, 1, \dots, k-2\}$ and for all $x_1, \dots, x_n \in \{0, 1, \dots, j\}$, we have that $f(x_1, \dots, x_n) \in \{0, 1, \dots, j\}$ and $g(x_1, \dots, x_n) = f(x_1, \dots, x_n)$. Let $a_1, \dots, a_n \in \{0, 1, \dots, j+1\}$. Then $f(a_1, \dots, a_n) \leftarrow f(a_1^{\leftarrow}, \dots, a_n^{\leftarrow}) \in \{0, 1, \dots, j\}$. Therefore $f(a_1, \dots, a_n) \in \{0, 1, \dots, j+1\}$.

Note that by the definition of g, we have $g(a_1, \dots, a_n) \in \{0, 1, \dots, j+1\}$. Since $a_1^{\leftarrow}, \dots, a_n^{\leftarrow} \in \{0, 1, \dots, j\}$, by the induction hypothesis, $f(a_1^{\leftarrow}, \dots, a_n^{\leftarrow}) = g(a_1^{\leftarrow}, \dots, a_n^{\leftarrow})$. We have

$$g(a_1^{\leftarrow}, \dots, a_n^{\leftarrow}) = f(a_1^{\leftarrow}, \dots, a_n^{\leftarrow}) \rightarrow f(a_1, \dots, a_n)$$

$$\text{and } g(a_1^{\leftarrow}, \dots, a_n^{\leftarrow}) \rightarrow g(a_1, \dots, a_n)$$

We distinguish 4 cases. Case 1: $g(a_1^{\leftarrow}, \dots, a_n^{\leftarrow}) = 0$. Then $f(a_1, \dots, a_n) = 0 = g(a_1, \dots, a_n)$.

Case 2: $g(a_1^{\leftarrow}, \dots, a_n^{\leftarrow}) = 1$. Suppose to the contrary that $f(a_1, \dots, a_n) \neq g(a_1, \dots, a_n)$, then either $f(a_1, \dots, a_n) = 1$ and $g(a_1, \dots, a_n) = 2$, or $f(a_1, \dots, a_n) = 2$ and $g(a_1, \dots, a_n) = 1$. If $f(a_1, \dots, a_n) = 1$ and $g(a_1, \dots, a_n) = 2$, then using Proposition 2, we have

$$2 = g(a_1, \dots, a_n) \rightarrow g(a_1^{\rightarrow}, \dots, a_n^{\rightarrow}) \rightarrow \dots \rightarrow g(a_1^{(k-2)\rightarrow}, \dots, a_n^{(k-2)\rightarrow}) \in \{0, 1\}$$

Therefore $g(a_1^{(k-2)\rightarrow}, \dots, a_n^{(k-2)\rightarrow}) = 0$. Also,

$$1 = f(a_1, \dots, a_n) \rightarrow f(a_1^{\rightarrow}, \dots, a_n^{\rightarrow}) \rightarrow \dots \rightarrow f(a_1^{(k-2)\rightarrow}, \dots, a_n^{(k-2)\rightarrow}) \in \{0, 1\}$$

a chain of length $k-2$. Therefore $f(a_1^{(k-2)\rightarrow}, \dots, a_n^{(k-2)\rightarrow}) = 1$. By the definition of g, $1 = f(a_1^{(k-2)\rightarrow}, \dots, a_n^{(k-2)\rightarrow}) = g(a_1^{(k-2)\rightarrow}, \dots, a_n^{(k-2)\rightarrow}) = 0$, a contradiction. Similarly, it is impossible that $f(a_1, \dots, a_n) = 2$ and $g(a_1, \dots, a_n) = 1$. Therefore $f(a_1, \dots, a_n) = g(a_1, \dots, a_n)$.

Case 3: $g(a_1^{\leftarrow}, \dots, g_n^{\leftarrow}) = a \in \{2, \dots, k-2\}$. Then $g(a_1, \dots, a_n) = a + 1 = f(a_1, \dots, a_n)$.

Case 4: $g(a_1^{\leftarrow}, \dots, g_n^{\leftarrow}) = k - 1$. Then $g(a_1, \dots, a_n) = 0 = f(a_1, \dots, a_n)$.

In all cases, $g(a_1, \dots, a_n) = f(a_1, \dots, a_n)$. By induction, $f = g$. Therefore, $f \in \langle c_0, c_1, \min, \max \rangle$. □

By Claim 1 and Claim 3, $\operatorname{Pol} \sigma = \langle c_0, c_1, \max, \min \rangle$. □

When $A = \{0, 1\}$,

$$\begin{aligned} \operatorname{Sta}\langle c_0, c_1 \rangle &= \operatorname{Sta}\{c_0, c_1, e\} \\ &= \operatorname{Pol}\{(0,0), (1,1), (0,1)\} \\ &= \operatorname{Pol}\{(0,0), (1,1), (1,0)\} \\ &= \operatorname{Pol} \sigma \end{aligned}$$

Therefore $\langle c_0, c_1, \max, \min \rangle = \operatorname{Pol} \sigma$ is the largest clone in the interval. This was already known [4].

For $A = \{0, 1, 2\}$, we know that $\langle c_0, c_1, \max, \min \rangle$ is the largest clone in the interval $\operatorname{Int}\langle c_0, c_1 \rangle$ [2]. Given that $\langle c_0, c_1, \max, \min \rangle = \operatorname{Pol} \sigma$, we can also prove this result easily using the following theorem.

Theorem 9 (Bodnarčuk, Kalužnin, Kotov, Romov [7]). *Let A be a finite set. Let $\rho \subseteq A^h$, and let $\sigma \subseteq A^l$ be a relation without repetitions of coordinates. Then* $\operatorname{Pol}\rho \subseteq \operatorname{Pol}\sigma$ *iff there exist $m \geq l$, $n < m^h$ and an $n \times h$ matrix $X = (x_{i,j})$ with $x_{i,j} \in \{1, \ldots, m\}$ such that*

$$(a_1, \ldots, a_l) \in \sigma$$
$$\text{iff there exist } a_{l+1}, \ldots, a_m \text{ such that}$$
$$\text{for all } i = 1, \ldots, n, (a_{x_{i,1}}, a_{x_{i,2}}, \ldots, a_{x_{i,h}}) \in \rho$$

Corollary 10 (See [2]). *On $A = \{0, 1, 2\}$, the clone $\langle c_0, c_1, \max, \min\rangle$ is the largest clone in the interval* $\operatorname{Int}\langle c_0, c_1\rangle$.

Proof. By Theorem 5, we know that

$$\langle c_0, c_1, \max, \min\rangle = \operatorname{Pol}\{(0,0), (1,1), (1,2), (2,0)\}$$

The largest clone of the interval $\operatorname{Int}\langle c_0, c_1\rangle$ is the stabilizer

$$\operatorname{Sta}\langle c_0, c_1\rangle = \operatorname{Sta}\{c_0, c_1, e\} = \operatorname{Pol}\{(0,0,0), (1,1,1), (0,1,2)\}$$

By Theorem 9, using the matrix $X = \begin{pmatrix} 3 & 4 & 1 \\ 5 & 3 & 2 \end{pmatrix}$, we obtain

$$\operatorname{Pol}\{(0,0,0), (1,1,1), (0,1,2)\} \subseteq \operatorname{Pol}\{(0,0), (1,1), (1,2), (2,0)\}$$

By Lemma 8, we know that $\operatorname{Pol}\sigma \in \operatorname{Int}\langle c_0, c_1\rangle$. Therefore $\operatorname{Pol}\sigma = \operatorname{Sta}\langle c_0, c_1\rangle$, which implies that $\langle c_0, c_1, \max, \min\rangle$ is the largest clone in the interval $\operatorname{Int}\langle c_0, c_1\rangle$. □

For universes of more than 3 elements, it is no longer true that $\operatorname{Sta}\langle c_0, c_1\rangle = \langle c_0, c_1, \max, \min\rangle$. For example, for $A = \{0, 1, 2, 3\}$, the function $\max'$ defined according to the chain $0 < 2 < 3 < 1$ is in $\operatorname{Sta}\langle c_0, c_1\rangle$ but $\max' \notin \langle c_0, c_1, \max, \min\rangle$ where min and max are defined as in Theorem 5 (according to the chain $0 < 3 < 2 < 1$).

5. **The structure of the interval**

So far, we know what the monoidal interval $\operatorname{Int}\langle c_0, c_1\rangle$ looks like on 2-element and 3-element sets. They can be found in Figure 6 (where min and max are defined according to the ordering $0 < 2 < 1$).

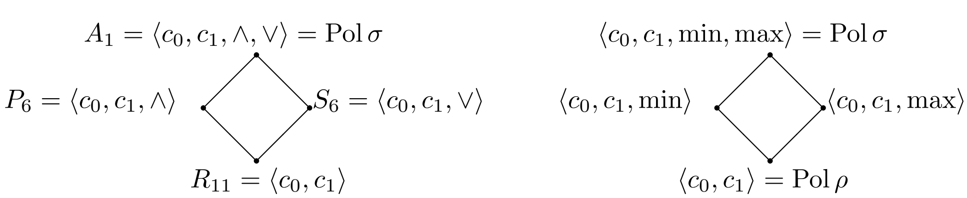

FIGURE 6. The intervals on 2-element and 3-element sets

We want to know what $\operatorname{Int}\langle c_0, c_1\rangle$ looks like in general. This is very difficult since $|\operatorname{Int}\langle c_0, c_1\rangle| = 2^{\aleph_0}$. But, we will present some of the structure near the bottom of the interval.

Proposition 11. *Let $\langle A; \wedge, \vee, c_0, c_1\rangle$ be a finite lattice with top element 1 and bottom element 0 such that $|A| \geq 2$. Then the clone $\langle c_0, c_1, \wedge, \vee\rangle$ is in the monoidal interval* $\mathrm{Int}\langle c_0, c_1\rangle$ *for the universe A.*

Proof. Straightforward. □

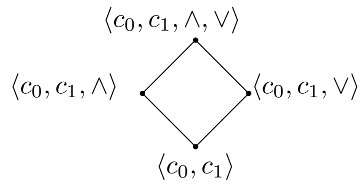

FIGURE 7. A subinterval of $\mathrm{Int}\langle c_0, c_1\rangle$ for distributive lattices of k elements

Theorem 12. *Let $\langle A; \wedge, \vee, c_0, c_1\rangle$ be a finite distributive lattice with top element 1 and bottom element 0 such that $|A| \geq 2$. Then $[\langle c_0, c_1\rangle, \langle c_0, c_1, \wedge, \vee\rangle]$ is the interval in Figure 7.*

Note that for $|A| = 2$, this was proved by Post [4]. For $|A| = 3$, it was proved by Krokhin [2] since the only possible lattice in this case is $\langle A; \min, \max, c_0, c_1\rangle$, which gives the result stated above.

Proof of Theorem 12. Let $|A| \geq 2$. Let $f \in \langle c_0, c_1, \wedge, \vee\rangle$ be n-ary and not constant. There exists a non-void and inclusion-free family $\mathcal{F}$ of non-void subsets of $N = \{1, \ldots, n\}$ (i.e. $X \not\subseteq Y$ for all $X, Y \in \mathcal{F}$) such that for all $a_1, \ldots, a_n \in A$

$$f(a_1, \ldots, a_n) = \bigvee_{X \in \mathcal{F}} (\bigwedge_{i \in X} a_i)$$

If $\mathcal{F} = \{\{i\}\}$, then $f = e_i^n$.

Claim 1. *If there exists $X \in \mathcal{F}$ with $|X| \geq 2$, then $\wedge \in \langle c_0, c_1, f\rangle$.*

Proof. For notational simplicity, let $X = \{1, 2, \ldots, i\}$ where $i \geq 2$. Form $g(x_1, x_2) \approx f(x_1, x_2, c_1, \ldots, c_1, c_0, \ldots, c_0)$ where the first c_0 is in the $(i+1)$-st place. Given that $Y \subset X$ for no $Y \in \mathcal{F}$, clearly $g = \wedge$. □

Claim 2. *If $|\mathcal{F}| \geq 2$, then $\vee \in \langle c_0, c_1, f\rangle$.*

Proof. Choose distinct $X, Y, \in \mathcal{F}$ with minimum $|X \cup Y|$. For notational simplicity, let $1 \in X \setminus Y$, $2 \in Y \setminus X$ and $X \cup Y = \{1, 2, \ldots, i\}$ where $i \geq 2$.

We claim that every $Z \in \mathcal{F}$ such that $Z \subseteq X \cup Y$ satisfies $1, 2 \in Z$. Indeed suppose to the contrary that $Z \subseteq (X \cup Y) \setminus \{1\}$. Then $1 \notin Z$ and $|Y \cup Z| \leq i - 1$ contrary to the minimality of i. Similarly, $|X \cup Z|$ is contrary to the minimality of i if $Z \subseteq (X \cup Y) \setminus \{2\}$. Set $h(x_1, x_2) \approx f(x_1, x_2, c_1, \ldots c_1, c_0, \ldots, c_0)$ where the first c_0 is at the $(i+1)$-st place. Now $h(x_1, x_2) \approx x_1 \vee x_2$ or $h(x_1, x_2) \approx x_1 \vee x_2 \vee (x_1 \wedge x_2)$ depending on if there exists such a Z. In the latter case, the absorption law yeilds $h(x_1, x_2) \approx x_1 \vee x_2$. □

Examining all the possible cases, we obtain exactly Figure 7. □

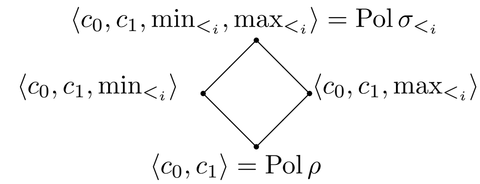

FIGURE 8. General min-max subintervals

Theorem 13. *Let $A = \{0, \dots, k-1\}$ and $k \geq 3$. Let $\{<_i \mid i \in \{0, \dots, (k-2)!\}\}$ be all the possible chains of A such that 0 is the smallest and 1 the largest elements in the chain. Define*

$$\rho = \{(0,0), (0,1), (1,1), (1,2), (2,3), \dots, (k-1, 0)\}$$

and for each $i \in \{0, \dots, (k-2)!\}$, define

$$\sigma_{<_i} = \{(0,0), (1,1), (a_2, 0), (a_3, a_2), \dots, (1, a_{k-1})\}$$

where $0 <_i a_2 <_i a_3 <_i \dots <_i a_{k-1} <_i 1$, and define $\max_{<_i}$ and $\min_{<_i}$ according to the ordering $<_i$. Then for each $<_i$, the interval $[\operatorname{Pol}\rho, \operatorname{Pol}\sigma_{<_i}]$ is contained in the monoidal interval $\operatorname{Int}\langle c_0, c_1\rangle$ and has Figure 8 as its Hasse diagram.

Proof. Follows from Theorems 1, 5 and 12, and Proposition 11. □

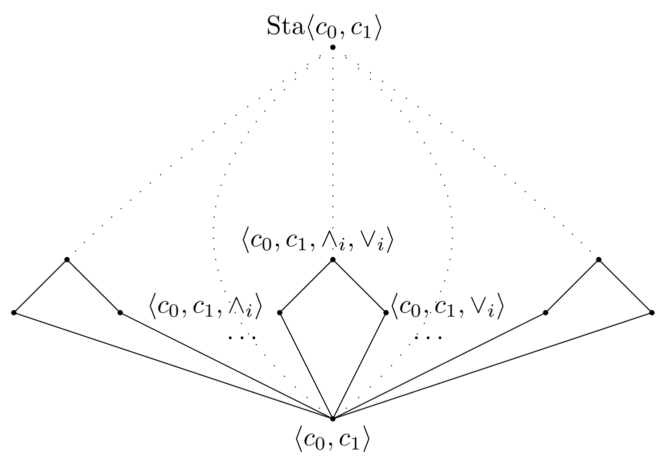

FIGURE 9. The interval $\operatorname{Int}\langle c_0, c_1\rangle$ for finite universes

Theorem 14. *Let $A = \{0, \dots, k-1\}$ and $k \geq 4$. Consider all the possible distributive lattices on A with top element 1 and bottom element 0:*

$$\{\langle A; \wedge_i, \vee_i, c_0, c_1\rangle \mid i \in I\}$$

Then the lower part of the monoidal interval $\operatorname{Int}\langle c_0, c_1\rangle$ contains the diamond shaped intervals $[\langle c_0, c_1\rangle, \langle c_0, c_1, \wedge_i, \vee_i\rangle]$ as shown in Figure 9.

Proof. Follows from Proposition 11 and Theorem 12. □

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