

**CLONES ON THREE ELEMENTS  
PRESERVING A BINARY RELATION  
(COMPLETE VERSION)**

ANNE FEARNLEY

ABSTRACT. We describe the clones on 3 elements that can be expressed as  $\text{Pol } \rho$  for  $\rho$  a binary relation. We present the poset of these clones ordered by inclusion.

1. INTRODUCTION

In 1941, Post presented the complete description of the countably many clones on 2 elements [9]. The structure of the lattice of clones on finitely many (but more than 2) elements is more complex; in fact the lattice is of cardinality  $2^{\aleph_0}$  [7]. It is hoped that by studying clones on 3 elements, we might get an idea of the general structure of the lattice of clones on any finite set of cardinality greater than 2.

It is also known that every clone  $C$  on a set  $A$  can be expressed as the clone of those operations preserving a set of relations on  $A$ , i.e.  $C = \text{Pol } R$  where  $R$  is a set of relations on  $A$ . This may be rewritten as  $C = \bigcap_{\rho \in R} \text{Pol } \rho$ . Hence those clones of operations preserving a single relation may be viewed as a sort of skeleton for the whole lattice  $\mathcal{L}_A$ . If we consider only the clones of operations preserving a single binary relation on 3 elements, we already have 266 clones in 67 equivalence classes. Included among them are 16 of the 18 maximal clones.

The author's Master's thesis was a compilation and an ordering by inclusion of the clones on three elements which preserve one binary relation. This article is a translation and a rewriting of this thesis. Errors have been corrected. Several inclusions have been simplified, in part due to Theorem 4 which is new. Some of the diagrams have been redrawn for clarity and a couple have been added. The non-inclusions section of the thesis has been omitted for brevity and because it is straightforward.

2. DEFINITIONS

Let  $A$  be a finite set and  $n$  a positive integer. An  $n$ -ary operation on  $A$  is a function  $f : A^n \rightarrow A$ . The set of all  $n$ -ary operations on  $A$  is denoted by  $\mathcal{O}_A^{(n)}$ , and  $\mathcal{O}_A := \bigcup_{0 < n < \omega} \mathcal{O}_A^{(n)}$ . For  $1 \leq i \leq n$ , the  $n$ -ary  $i$ -th projection is defined as  $e_i^{(n)}(x_1, \dots, x_n) = x_i$  for all  $x_1, \dots, x_n$ . We write  $e$  for the identity operation. For  $a \in A$ , the  $n$ -ary constant operation  $a$  is defined as  $c_a^{(n)}(x_1, \dots, x_n) = a$  for all  $x_1, \dots, x_n$ . We write simply  $c_a$  for unary constant operations.

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For  $f \in \mathcal{O}^{(n)}$ , and  $g_1, \dots, g_n \in \mathcal{O}^{(m)}$ , we define their composition to be the  $m$ -ary operation  $f[g_1, \dots, g_n]$  defined by

$$f[g_1, \dots, g_n](x_1, \dots, x_m) = f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m))$$

A *clone* on  $A$  is a subset  $F$  of  $\mathcal{O}_A$  which contains all the projections and is closed under composition. It is well known and easy to prove that the intersection of an arbitrary set of clones on  $A$  is a clone on  $A$ . Thus for  $F \subseteq \mathcal{O}_A$ , there exists the least clone containing  $F$ , called the clone *generated* by  $F$  and denoted by  $\langle F \rangle$ . Equivalently,  $\langle F \rangle$  is the set of *term operations* of the algebra  $\mathbf{A} = \langle A; F \rangle$  usually denoted by  $T(\mathbf{A})$ . The clones on  $A$ , ordered by inclusion, form the complete lattice  $\mathcal{L}_A$ .

Let  $h$  be a positive integer. A  *$h$ -ary relation*  $\rho$  is a subset of  $A^h$ . For  $\rho \in A^2$ , we write  $a \rightarrow b$  for  $(a, b) \in \rho$ . The relations may then be drawn as directed graphs. For example, for  $A = \{0, 1, 2\}$ , the relation  $\{(0, 0), (0, 1), (1, 0), (1, 2)\}$  may be represented as in Figure 1:



FIGURE 1. Example of a relation

Let  $f \in \mathcal{O}^{(n)}$ , and let  $\rho$  be an  $h$ -ary relation on  $A$ . The operation  $f$  *preserves*  $\rho$  if for all  $(a_{1,i}, a_{2,i}, \dots, a_{h,i}) \in \rho$  ( $i = 1, \dots, n$ ),

$$(f(a_{1,1}, a_{1,2}, \dots, a_{1,n}), f(a_{2,1}, a_{2,2}, \dots, a_{2,n}), \dots, f(a_{h,1}, a_{h,2}, \dots, a_{h,n})) \in \rho$$

The set of operations on  $A$  preserving  $\rho$  is a clone denoted by  $\text{Pol } \rho$ . A relation  $\rho$  is *strongly rigid* if it is preserved only by the projections.

From now on, we will assume that we are working on the set  $\mathbf{3} := \{0, 1, 2\}$  and that relations are binary. When we draw them as directed graphs, we will omit the numbers; they will be assumed to be in the same configuration as in Figure 1. Note that the unary relations are equivalent to certain binary relations. For example, the unary relation  $\{0, 1\}$  can be represented as  $\{(0, 0), (1, 1)\}$ , so that we are in fact studying all unary relations as well.

### 3. DESCRIPTION OF SOME CLONES

It is easy to see that the trivial relations  $\emptyset$  and  $\mathbf{3}^n$  are preserved by all operations. For binary relations, that means that  $\cdot \cdot$ ,  $\triangleleft$  and  $\circ \circ$  (because it corresponds to the full unary relation) are preserved by all operations.

The following four theorems exhibit relations for which we know all the operations which preserve them.

**Theorem 1** (Rosenberg [10]). *The relation  $\triangleleft$  is strongly rigid.*

**Theorem 2** (Fearnley [3]). *The relation  $\triangleleft$  is strongly rigid.*

**Theorem 3** (Fearnley [5]).  $\text{Pol}(\triangleleft) = \langle c_0, c_1 \rangle$ .

**Theorem 4.** Let  $A = \{0, 1, \dots, k-1\}$  where  $k > 2$ . Let  $\rho$  be the following binary relation on  $A$  (see Figure 2)

$$\rho = \{(0, 0), (0, 1), (1, 2), \dots, (k-2, k-1), (k-1, 1)\}$$

Then  $\text{Pol } \rho = \langle c_0, \text{maj}_0 \rangle$  where  $\text{maj}_0$  is the ternary operation defined by

$$\text{maj}_0(x, y, z) := \begin{cases} i, & \text{if } x = y = i \text{ or } x = z = i \text{ or } y = z = i; \\ 0, & \text{otherwise.} \end{cases}$$

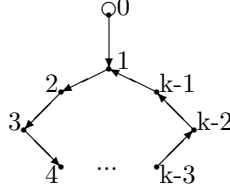


FIGURE 2

Before proving the theorem, let us state a pair of definitions with some of their properties, and prove a lemma. We define two unary operations:  $x^{\rightarrow}$  given by  $0^{\rightarrow} = 0$ ,  $a^{\rightarrow} = a + 1$  if  $0 < a < k - 1$  and  $(k - 1)^{\rightarrow} = 1$ , and  $x^{\leftarrow}$  given by  $0^{\leftarrow} = 0$  and  $a^{\leftarrow} = a - 1$  if  $0 < a \leq k - 1$ . We write  $a^{2^{\rightarrow}}$  instead of  $a^{\rightarrow^{\rightarrow}}$ , and so on. We define  $a^{i^{\leftarrow}}$  similarly.

- Proposition 5.**
- (A)  $a \rightarrow a^{\rightarrow}$  and  $a^{(k-1)^{\rightarrow}} = a$  for all  $a \in A$ .
  - (B)  $a^{\leftarrow} \rightarrow a$ ,  $a^{(k-2)^{\leftarrow}} \in \{0, 1\}$  and  $a^{(k-1)^{\leftarrow}} = 0$  for all  $a \in A$ .
  - (C) Either  $\{a, a^{\rightarrow}, a^{2^{\rightarrow}}, \dots, a^{(k-2)^{\rightarrow}}\} = \{1, 2, \dots, k-1\}$   
or  $a = a^{\rightarrow} = a^{2^{\rightarrow}} = \dots = a^{(k-2)^{\rightarrow}} = 0$ .
  - (D) Let  $f \in \text{Pol } \rho$  be an  $n$ -ary operation. Then  $f(x_1^{\rightarrow}, \dots, x_n^{\rightarrow}) = (f(x_1, \dots, x_n))^{\rightarrow}$  for all  $x_1, \dots, x_n \in A$ .

*Proof.* Statements (A) and (B) are trivial. Statement (C) is derived from statement (A) since  $a \rightarrow a^{\rightarrow} \rightarrow a^{2^{\rightarrow}} \rightarrow \dots \rightarrow a^{(k-1)^{\rightarrow}} = a$ .

To prove statement (D), let  $x_1, \dots, x_n \in A$ . By (A), we have that

$$(1) \quad f(x_1, \dots, x_n) \rightarrow f(x_1, \dots, x_n)^{\rightarrow}$$

Since  $f \in \text{Pol } \rho$ , and since by (A),  $x_1 \rightarrow x_1^{\rightarrow}, \dots, x_n \rightarrow x_n^{\rightarrow}$ , it follows that

$$(2) \quad f(x_1, \dots, x_n) \rightarrow f(x_1^{\rightarrow}, \dots, x_n^{\rightarrow})$$

If  $f(x_1, \dots, x_n) \neq 0$ , then  $f(x_1^{\rightarrow}, \dots, x_n^{\rightarrow})$  is uniquely determined, so by equations 1 and 2,  $f(x_1^{\rightarrow}, \dots, x_n^{\rightarrow}) = (f(x_1, \dots, x_n))^{\rightarrow}$  as required. If  $f(x_1, \dots, x_n) = 0$  and  $f(x_1^{\rightarrow}, \dots, x_n^{\rightarrow}) = 0$ , then the equality is fulfilled. However, if  $f(x_1, \dots, x_n) = 0$  and  $f(x_1^{\rightarrow}, \dots, x_n^{\rightarrow}) = 1$ , then from 2, it follows that  $1 = f(x_1^{\rightarrow}, \dots, x_n^{\rightarrow}) \rightarrow f(x_1^{2^{\rightarrow}}, \dots, x_n^{2^{\rightarrow}}) \rightarrow \dots \rightarrow f(x_1^{(k-1)^{\rightarrow}}, \dots, x_n^{(k-1)^{\rightarrow}}) = f(x_1, \dots, x_n) = 0$  which is a contradiction.  $\square$

**Lemma 6.** Let  $f \in \text{Pol } \rho$  be an  $n$ -ary operation, and let  $x_1, \dots, x_n \in \{0, 1\}$ . Then  $f(0, \dots, 0) = 0$  and  $f(x_1, \dots, x_n) \in \{0, 1\}$ .

*Proof.* Since  $0 \leftrightarrow 0$ , we have  $f(0, \dots, 0) \leftrightarrow f(0, \dots, 0)$ . Therefore  $f(0, \dots, 0) = 0$ . Now  $0 \rightarrow x_i$  for  $i = 1, \dots, n$ , thus  $0 = f(0, \dots, 0) \rightarrow f(x_1, \dots, x_n)$ . Therefore  $f(x_1, \dots, x_n) \in \{0, 1\}$ .  $\square$

**Proof of Theorem 4.** We begin by showing that  $\langle c_0, \text{maj}_0 \rangle \subseteq \text{Pol } \rho$ . Then, for an operation  $f \in \text{Pol } \rho$ , we consider its diagonal which being unary must be a constant or the identity. We show that a constant diagonal implies that the original operation is a constant. When the diagonal is the identity, we consider the restriction of  $f$  to  $\{0, 1\}$ . The Boolean clones are all known [9]. We find that  $f|_{\{0,1\}} \subseteq \langle c_0, \text{maj} \rangle$  on  $\{0, 1\}$ . Finally, we show that what happens on  $\{0, 1\}$  determines what happens on the whole set. This completes the proof.

**Claim 1.**  $c_0$  and  $\text{maj}_0$  are in  $\text{Pol } \rho$ .

*Proof.* For  $c_0$ , note that if  $a \rightarrow b$  then  $c_0(a) = 0 \rightarrow 0 = c_0(b)$ . Therefore  $c_0 \in \text{Pol } \rho$ .

For  $\text{maj}_0$ , let  $a_1, a_2, b_1, b_2, c_1, c_2 \in A$  such that  $a_1 \rightarrow a_2$ ,  $b_1 \rightarrow b_2$  and  $c_1 \rightarrow c_2$ . If  $a_1 = b_1 \neq 0$ , then  $a_2 = a_1 \rightarrow = b_2$ . Thus  $\text{maj}_0(a_1, b_1, c_1) = \text{maj}_0(a_1, a_1, c_1) = a_1 \rightarrow a_1 \rightarrow = a_2 = \text{maj}_0(a_2, a_2, c_2) = \text{maj}_0(a_2, b_2, c_2)$  as required. The cases where  $a_1 = c_1 \neq 0$  and  $b_1 = c_1 \neq 0$  are similar. In all other cases,  $\text{maj}_0(a_1, b_1, c_1) = 0$ , and the only way we could fail to have  $\text{maj}_0(a_1, b_1, c_1) \rightarrow \text{maj}_0(a_2, b_2, c_2)$  would be if  $\text{maj}_0(a_2, b_2, c_2) \notin \{0, 1\}$ . For that to happen, we would have to have  $a_2 = b_2 \notin \{0, 1\}$  or  $a_2 = c_2 \notin \{0, 1\}$  or  $b_2 = c_2 \notin \{0, 1\}$  which have already been covered in a previous case. Therefore  $\text{maj}_0 \in \text{Pol } \rho$ .  $\square$

Let  $f \in \text{Pol } \rho$  be an  $n$ -ary operation. Define its diagonal operation  $d : A \rightarrow A$  by  $d(x) := f(x, \dots, x)$  for all  $x \in A$ .

**Claim 2.** Either  $d = c_0$  or  $d = e$ .

*Proof.* By Lemma 6,  $d(0) = f(0, \dots, 0) = 0$  and  $d(1) \in \{0, 1\}$ . If  $d(1) = 0$ , we have  $d(2) \rightarrow \dots \rightarrow d(k-1) \rightarrow d(1) = 0$ . Therefore  $d(2) = \dots = d(k-1) = 0$ , which implies that  $d = c_0$ . If  $d(1) = 1$ , we have  $1 = d(1) \rightarrow d(2) \rightarrow \dots \rightarrow d(k-1) \rightarrow d(1) = 1$ . Therefore  $d(a) = a$  for all  $a \in A$ , which implies that  $d = e$ .  $\square$

**Claim 3.** If  $d = c_0$  then  $f = c_0^{(n)}$ .

*Proof.* Let  $x_1, \dots, x_n \in \{0, 1\}$ , then  $x_i^{(k-2)\rightarrow} \in \{k-1, 0\}$  and hence  $x_i^{(k-2)\rightarrow} \rightarrow 1$  for all  $i = 1, \dots, n$ . By Propositions 5A and 5D, we obtain  $f(x_1, \dots, x_n) \rightarrow f(x_1 \rightarrow, \dots, x_n \rightarrow) \rightarrow \dots \rightarrow f(x_1^{(k-2)\rightarrow}, \dots, x_n^{(k-2)\rightarrow}) \rightarrow f(1, \dots, 1) = d(1) = 0$ . Therefore  $f(x_1, \dots, x_n) = 0$  for all  $x_1, \dots, x_n \in \{0, 1\}$ . Now let  $x_1, \dots, x_n \in A$ . By Propositions 5A, 5D and the result above, we have  $f(x_1, \dots, x_n) \leftarrow f(x_1 \leftarrow, \dots, x_n \leftarrow) \leftarrow \dots \leftarrow f(x_1^{(k-2)\leftarrow}, \dots, x_n^{(k-2)\leftarrow}) = 0$ . Thus  $f(x_1, \dots, x_n) \in \{0, \dots, k-2\}$ . Similarly,  $(f(x_1, \dots, x_n)) \rightarrow = f(x_1 \rightarrow, \dots, x_n \rightarrow) \in \{0, \dots, k-2\}$ . In the same way, we find that  $(f(x_1, \dots, x_n))^{2\rightarrow}, \dots, (f(x_1, \dots, x_n))^{(k-2)\rightarrow} \in \{0, \dots, k-2\}$ . By Proposition 5C, this implies that  $f(x_1, \dots, x_n) = 0$ .  $\square$

For every  $f : A^n \rightarrow A$  in  $\text{Pol } \rho$ , we consider the corresponding Boolean operation  $f|_{\{0,1\}} : \{0, 1\}^n \rightarrow \{0, 1\}$ . This is possible because of Lemma 6. Note that  $\text{maj}_0|_{\{0,1\}} = \text{maj}$  (the usual Boolean majority operation) and that  $c_0$  become the corresponding Boolean constant  $c_0$ . Now define  $(\text{Pol } \rho)|_{\{0,1\}} := \{f|_{\{0,1\}} \mid f \in \text{Pol } \rho\}$ . Clearly,  $(\text{Pol } \rho)|_{\{0,1\}}$  is a clone on  $\{0, 1\}$ .

**Claim 4.**  $(\text{Pol } \rho)|_{\{0,1\}} = \langle c_0, \text{maj} \rangle$ .

*Proof.* Using Post's classification [9], we can find what  $(\text{Pol } \rho)|_{\{0,1\}}$  is. By Claim 2, we need only consider the clones on  $\{0, 1\}$  which contain  $c_0$ , but not  $c_1$ . These

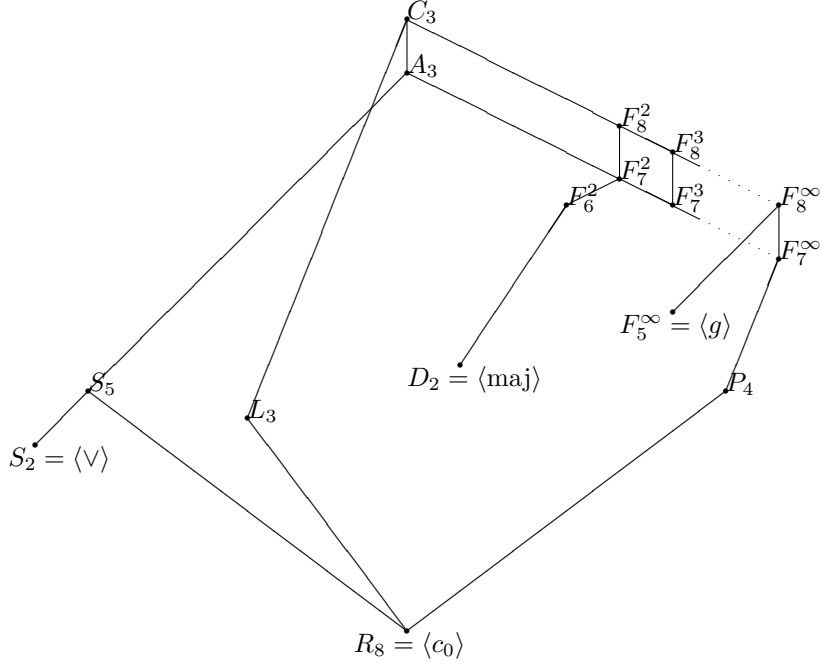


FIGURE 3. A part of Post's lattice

clones are shown in Figure 3, along with the clones  $F_5^\infty$ ,  $D_2$  and  $S_2$  which are referred to in this proof, and a few other clones needed to situate them.

Since  $c_0$  and  $\text{maj}$  are in  $(\text{Pol } \rho)|_{\{0,1\}}$ , that means that  $\langle c_0 \rangle = R_8$  and  $\langle \text{maj} \rangle = D_2$  are included in  $(\text{Pol } \rho)|_{\{0,1\}}$ , hence  $F_7^2 = \langle c_0, \text{maj} \rangle \subseteq (\text{Pol } \rho)|_{\{0,1\}}$ .

But the operation  $g(x, y, z) := x \wedge (y \vee \neg z) \notin (\text{Pol } \rho)|_{\{0,1\}}$  since otherwise, there would be an operation  $f \in \text{Pol } \rho$  such that  $f|_{\{0,1\}}(x, y, z) = x \wedge (y \vee \neg z)$ . Such an operation would have the property that  $1 = f(1, 0, 0) \rightarrow f(2, 0, 0) \rightarrow \dots \rightarrow f(k-1, 0, 0) \rightarrow f(1, 0, 1) = 0$ , which is impossible. Therefore  $\langle g \rangle = F_5^\infty \not\subseteq (\text{Pol } \rho)|_{\{0,1\}}$ .

The operation  $x \vee y \notin (\text{Pol } \rho)|_{\{0,1\}}$ , since otherwise, there would be a operation  $f \in \text{Pol } \rho$  such that  $f|_{\{0,1\}}(x, y) = x \vee y$ . Such an operation would have the property that

$$1 = f(0, 1) \rightarrow f(1, 2) \rightarrow \dots \rightarrow f(k-3, k-2) \rightarrow f(k-2, k-1) \rightarrow f(k-1, 1)$$

which implies that  $f(k-1, 1) = 1$ , and

$$1 = f(1, 0) \rightarrow f(2, 0) \rightarrow \dots \rightarrow f(k-2, 0) \rightarrow f(k-1, 1) = 1$$

which is impossible because it is only a  $(k-2)$ -cycle. Therefore  $\langle \vee \rangle = S_2 \not\subseteq (\text{Pol } \rho)|_{\{0,1\}}$ . Hence  $(\text{Pol } \rho)|_{\{0,1\}} = F_7^2 = \langle c_0, \text{maj} \rangle$ .  $\square$

**Claim 5.** *Let  $f, g \in \text{Pol } \rho$  such that  $f|_{\{0,1\}} = g|_{\{0,1\}}$ . Then  $f = g$ .*

*Proof.* Let  $f, g$  be  $n$ -ary and let  $x_1, \dots, x_n \in A$ . By Proposition 5, we have for all  $i \in \{0, \dots, k-2\}$ ,

$$\begin{aligned} (g(x_1, \dots, x_n))^{i \rightarrow} &= g(x_1^{i \rightarrow}, \dots, x_n^{i \rightarrow}) \leftarrow g((x_1^{i \rightarrow})^{\leftarrow}, \dots, (x_n^{i \rightarrow})^{\leftarrow}) \leftarrow \dots \\ &\leftarrow g((x_1^{i \rightarrow})^{(k-2)\leftarrow}, \dots, (x_n^{i \rightarrow})^{(k-2)\leftarrow}) = f((x_1^{i \rightarrow})^{(k-2)\leftarrow}, \dots, (x_n^{i \rightarrow})^{(k-2)\leftarrow}) \\ &\rightarrow \dots \rightarrow f((x_1^{i \rightarrow})^{\leftarrow}, \dots, (x_n^{i \rightarrow})^{\leftarrow}) \rightarrow f(x_1^{i \rightarrow}, \dots, x_n^{i \rightarrow}) = (f(x_1, \dots, x_n))^{i \rightarrow} \end{aligned}$$

Furthermore, by Lemma 6,  $f((x_1^{i\rightarrow})^{(k-2)\leftarrow}, \dots, (x_n^{i\rightarrow})^{(k-2)\leftarrow}) \in \{0, 1\}$ . If there exists an  $i$  such that  $f((x_1^{i\rightarrow})^{(k-2)\leftarrow}, \dots, (x_n^{i\rightarrow})^{(k-2)\leftarrow}) = 1$ , then we would have for that  $i$  that

$$(g(x_1, \dots, x_n))^{i\rightarrow} = k - 1 = (f(x_1, \dots, x_n))^{i\rightarrow}$$

which implies that

$$g(x_1, \dots, x_n) = (k - 1)^{((k-1)-i)\rightarrow} = f(x_1, \dots, x_n)$$

as required. Now suppose that  $f((x_1^{i\rightarrow})^{(k-2)\leftarrow}, \dots, (x_n^{i\rightarrow})^{(k-2)\leftarrow}) = 0$  for all  $i \in \{0, \dots, k - 2\}$ . In that case,  $(f(x_1, \dots, x_n))^{i\rightarrow}, (g(x_1, \dots, x_n))^{i\rightarrow} \in \{0, \dots, k - 2\}$ . Therefore, by Proposition 5,  $g(x_1, \dots, x_n) = 0 = f(x_1, \dots, x_n)$  as required.  $\square$

Thus  $f \in \text{Pol } \rho$  is entirely determined by  $f|_{\{0,1\}}$ . By Claim 4,  $f|_{\{0,1\}}$  can be written as a term made up of  $c_0$  and  $\text{maj}$  on  $\{0, 1\}$ . If we replace all occurrences of  $c_0$  and  $\text{maj}$  in the term by the corresponding operations on  $A$ , we must obtain  $f$ . Therefore  $f$  can be written as a term made up of  $\text{maj}_0$  and  $c_0$  on  $A$ . In other words,  $f \in \langle c_0, \text{maj}_0 \rangle$ . Therefore, by Claim 1,  $\text{Pol } \rho = \langle c_0, \text{maj}_0 \rangle$ .  $\square$

**Corollary 7.** For  $A = \{0, 1, 2\}$ ,  $\text{Pol}(\begin{smallmatrix} \circ \\ \swarrow \searrow \\ \rightarrow \end{smallmatrix}) = \text{Pol}(\begin{smallmatrix} \circ \\ \swarrow \searrow \\ \leftarrow \end{smallmatrix}) = \langle c_0, \text{maj}_0 \rangle$ .

#### 4. INCLUSIONS

For the relations not covered in Section 3, we must find out which clones are included in which others. The following theorem provides a way of finding inclusions directly from the relations. Note that a relation  $\rho \subseteq A^h$  is *without repetitions* if  $\rho \not\subseteq \{(a_1, \dots, a_h) \in A^h \mid a_i = a_j \text{ if } (i, j) \in \epsilon\}$  for any equivalence  $\epsilon \in \{1, \dots, h\}^2$  other than the trivial equivalence  $\{(1, 1), (2, 2), \dots, (h, h)\}$ . For binary relations on 3 elements, this means that  $\rho \not\subseteq \{(0, 0), (1, 1), (2, 2)\}$ . Also note that if  $\rho \subseteq \{(0, 0), (1, 1), (2, 2)\}$ , then  $\text{Pol } \rho = \text{Pol } \rho^{(1)}$  where  $\rho^{(1)} = \{a \mid (a, a) \in \rho\}$  which is without repetitions.

**Theorem 8** (Bodnarčuk, Kalužnin, Kotov, Romov [2]). *Let  $A$  be a finite set. Let  $\rho \subseteq A^h$ , and let  $\sigma \subseteq A^l$  be a relation without repetitions. Then  $\text{Pol } \rho \subseteq \text{Pol } \sigma$  iff there exist  $m \geq l$ ,  $n < m^h$  and an  $n \times h$  matrix  $X = (x_{ij})$  with  $x_{ij} \in \{1, \dots, m\}$  such that*







$$\begin{aligned} & (a_1, \dots, a_l) \in \sigma \\ & \text{iff there exist } a_{l+1}, \dots, a_m \text{ such that} \\ & \text{for all } i = 1, \dots, n, (a_{x_{i,1}}, a_{x_{i,2}}, \dots, a_{x_{i,h}}) \in \rho \end{aligned}$$

We can greatly decrease the number of relations we need to examine by considering the following results.

**Proposition 9.** *Let  $\pi$  be a permutation of  $A$ ,  $\rho$  an  $h$ -ary relation on  $A$  and  $f$  an  $n$ -ary operation on  $A$ . Set  $\pi(\rho) := \{(\pi(a_1), \dots, \pi(a_h)) \mid (a_1, \dots, a_h) \in \rho\}$ , and  $f_\pi : A^n \rightarrow A$  defined by  $f_\pi(x_1, \dots, x_n) = \pi(f(\pi^{-1}(x_1), \dots, \pi^{-1}(x_n)))$ . Then  $\text{Pol}(\pi(\rho)) = (\text{Pol } \rho)_\pi := \{f_\pi \mid f \in \text{Pol } \rho\}$ .*

**Corollary 10.** *Let  $\rho$  be a binary relation on  $A$ . Set  $\rho' = \{(b, a) \mid (a, b) \in \rho\}$ . Then  $\text{Pol } \rho = \text{Pol } \rho'$ .*

*Proof.* Use Theorem 8 with the matrix  $\begin{pmatrix} 1 & 0 \end{pmatrix}$ .  $\square$























**Theorem 11** (Jablonskij [6] and Rosenberg [11] and [12]). *There are 18 maximal clones in the lattice of clones on 3 elements. The 16 that are preserved by binary relations are of 6 types. These relations are: , , , , , and *

Theorem 8 gives a necessary and sufficient condition for inclusion. In practise, proving that there is no inclusion, using the theorem, is difficult. To show that the clone  $\text{Pol } \rho$  is not included in the clone  $\text{Pol } \sigma$ , we only need to find an operation  $f$  which preserves  $\rho$  but not  $\sigma$ . We have found such a function for each pair of relations  $\sigma, \rho$  for which  $\text{Pol } \rho \not\subseteq \text{Pol } \sigma$ . Finding such operations is straightforward: the operations used are at most 4-ary, and almost all are at most binary. The tables of operations will not be included in this paper, but they can be found in [4].

5. RELATIONS, EQUIVALENCES AND COVERINGS

As we saw in Proposition 9, permuting the elements in the universe yields relations with similar properties. The following lists the binary relations on 3 elements: one relation of each type. The relations are written as graphs (see Section 2). If a relation is preserved by exactly the same functions as one already named, the equivalent relation is noted.
















For each relation  $\rho$ , we indicate which relations  $\sigma$  are such that  $\text{Pol } \rho \subseteq \text{Pol } \sigma$  minimally (within the scope of this study), along with the matrix or theorem that proves it. The diagrams where these inclusions can be found is also shown.

<i>Cardinality 0.</i>					
				Greatest clone (Section 3)	
<i>Cardinality 1.</i>					
		$<$		Maximal clone	7.1, 7.4.2, 7.5, 7.9
		$<$	$\approx \{0\}$	(1 2)	7.5
		$<$	$\approx \{1\}$	(2 1)	7.5
<i>Cardinality 2.</i>					
		$<$	$\approx \{0\}$	(1 1)	7.1.1, 7.4.2, 7.5
		$<$	$\approx \{0, 1\}$	(2 1)	7.3.2, 7.4.2, 7.5
		$<$		Maximal clone	7.2, 7.3, 7.4, 7.5, 7.9
		$<$		$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$	7.1.1
		$<$	$\approx \{0\}$	(1 2)	7.5, 7.9
		$<$	$\approx \{1, 2\}$	(2 1)	7.5, 7.9
		$<$	$\approx \{0, 1\}$	(1 2)	7.9











	<		$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 3 & 1 \\ 4 & 2 \\ 5 & 4 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 1 & 3 \\ 3 & 4 \\ 5 & 2 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 1 & 3 \\ 4 & 2 \\ 5 & 4 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$	7.5

*Cardinality 3.*

	$\approx$		$\approx \{0\}$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \end{pmatrix}$	
	<			$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	7.1.1, 7.3.2
	<		$\approx \{0, 1\}$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$	7.2.1, 7.3.1, 7.4.1
	<			$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	7.1.1
	<			$\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}$	7.6.1
	<			$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.1.1
	<			$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$	7.1.1
	<		$\approx \{0, 2\}$	$\begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 3 & 2 \end{pmatrix}$	7.2.1, 7.3.1, 7.4.1
	<			$\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}$	7.6
	$\approx$				Section 3
	<		$\approx \{0\}$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$	7.1.2, 7.9













	<		$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	7.9
	<		$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.6.2
	<		$\begin{pmatrix} 3 & 1 \\ 3 & 4 \\ 4 & 2 \end{pmatrix}$	7.6.2, 7.9
	<		$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 3 & 1 \\ 3 & 4 \\ 4 & 2 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 3 & 4 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 4 & 3 \end{pmatrix}$	7.5
	<		$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$	7.5
	<		Maximal clone	7.10




























*Cardinality 4.*










	<		$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	7.1.1, 7.3.2
	<		$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$	7.2.1, 7.3.1, 7.4.1
	<		$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$	7.1.1
	$\approx$		$\approx \{0, 1\}$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$ and $(1)$
	<		$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	7.1.1, 7.3.2

$$\begin{array}{lcl}
 & & \begin{pmatrix} 3 & 1 \\ 3 & 2 \\ 4 & 1 \\ 4 & 5 \\ 5 & 2 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 3 & 1 \\ 3 & 4 \\ 4 & 2 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 3 & 2 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 3 & 1 \\ 3 & 4 \\ 4 & 2 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \\ 3 & 3 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 3 & 3 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 3 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 3 & 2 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 1 & 3 \\ 3 & 2 \\ 4 & 1 \\ 4 & 4 \end{pmatrix} \\
 & & \\
 & & \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 3 \\ 4 & 1 \end{pmatrix}
 \end{array}$$







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 7.7.1  
 7.3.2  
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 7.2.1, 7.3.1, 7.4.1  
 7.7.2  
 7.6.2  
 7.1.1  
 7.1.1, 7.6.1  
 7.1.1, 7.6.1  
 7.1.1  
 7.1.2  
 7.1.1, 7.3.2  
 7.1.1, 7.3.2

< 	$\left( \begin{array}{c} 1 \ 3 \\ 2 \ 4 \\ 3 \ 4 \\ 5 \ 1 \\ 5 \ 2 \end{array} \right)$	7.3.1, 7.4.1, 7.6.1
< 		7.3.1, 7.4.1, 7.6.1
< 	$\left( \begin{array}{c} 2 \ 3 \\ 3 \ 1 \\ 3 \ 4 \\ 4 \ 1 \end{array} \right)$	7.1.1
< 		7.1.1
< 	$\left( \begin{array}{c} 1 \ 3 \\ 1 \ 4 \\ 3 \ 2 \\ 4 \ 3 \end{array} \right)$	7.1.2
< 		7.3.2, 7.4.2
< 	$\left( \begin{array}{c} 2 \ 3 \\ 3 \ 1 \\ 3 \ 1 \\ 3 \ 4 \\ 4 \ 2 \end{array} \right)$	7.3.2, 7.4.2
< 		7.1.2
< 	$\left( \begin{array}{c} 1 \ 5 \\ 2 \ 6 \\ 3 \ 1 \\ 3 \ 8 \\ 4 \ 2 \\ 5 \ 3 \\ 5 \ 7 \\ 6 \ 4 \\ 7 \ 6 \\ 8 \ 4 \end{array} \right)$	7.3.2, 7.4.2
< 		7.3.2, 7.4.2
< 	$\left( \begin{array}{c} 1 \ 3 \\ 2 \ 4 \\ 3 \ 5 \\ 5 \ 4 \end{array} \right)$	7.7.1
< 		7.7.1

























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	<		Corollary 7	7.1.1
	<		Corollary 7	7.1.1
	<		Corollary 7	7.6.2
	<		Corollary 7	7.6.2
	<		Corollary 7	7.1, 7.3, 7.4, 7.6.1, 7.7.1
	<		Corollary 7	7.3.1, 7.4.1, 7.6.1, 7.7.2
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	<		Corollary 7	7.3.1, 7.4.1
	<		Corollary 7	7.1.2, 7.7.3, 7.9
	<		Corollary 7	7.7.2, 7.8
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	<		Corollary 7	7.1.1, 7.3.2, 7.7.1
	<		Corollary 7	7.1.1, 7.3.2, 7.7.1
	<		Corollary 7	7.3.1, 7.4.1, 7.6.1
	<		Corollary 7	7.3.1, 7.4.1, 7.6.1
	<		Corollary 7	7.3.2, 7.4.2
	<		Corollary 7	7.3.2, 7.4.2
	<		$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	7.2.1, 7.6
	<		$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$	7.2.1, 7.6
	<		$\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 4 \\ 5 & 1 \\ 5 & 2 \end{pmatrix}$	7.6.2
	<		$\begin{pmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \\ 4 & 5 \\ 5 & 2 \end{pmatrix}$	7.6.2






















<		$\left( \begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ 6 \\ 6 \end{array} \right. \left. \begin{array}{c} 1 \\ 3 \\ 4 \\ 4 \\ 1 \\ 2 \\ 5 \end{array} \right)$	7.2.2, 7.7.3
<		$\left( \begin{array}{c} 1 \\ 3 \end{array} \right. \left. \begin{array}{c} 3 \\ 2 \end{array} \right)$	7.2.1
<		$\left( \begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \end{array} \right. \left. \begin{array}{c} 1 \\ 3 \\ 4 \\ 4 \end{array} \right)$	7.2.1
<		$\left( \begin{array}{c} 1 \\ 3 \\ 4 \\ 4 \end{array} \right. \left. \begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \end{array} \right)$	7.2.1
<		$\left( \begin{array}{c} 3 \\ 3 \\ 4 \\ 4 \\ 5 \\ 6 \\ 6 \\ 6 \\ 7 \end{array} \right. \left. \begin{array}{c} 1 \\ 4 \\ 2 \\ 5 \\ 6 \\ 1 \\ 7 \\ 2 \end{array} \right)$	7.7.2, 7.8
<		$\left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 5 \\ 6 \\ 6 \end{array} \right. \left. \begin{array}{c} 3 \\ 4 \\ 4 \\ 1 \\ 2 \\ 5 \end{array} \right)$	7.7.3
<		$\left( \begin{array}{c} 1 \\ 4 \\ 4 \\ 5 \\ 5 \\ 6 \end{array} \right. \left. \begin{array}{c} 3 \\ 1 \\ 5 \\ 2 \\ 6 \\ 3 \end{array} \right)$	7.6.1, 7.7.2
<		$\left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 5 \\ 5 \\ 6 \end{array} \right. \left. \begin{array}{c} 3 \\ 4 \\ 4 \\ 2 \\ 6 \\ 3 \end{array} \right)$	7.6.1, 7.7.2
	<	$\left( \begin{array}{c} 1 \\ 1 \end{array} \right. \left. \begin{array}{c} 1 \\ 2 \end{array} \right)$	7.1.1, 7.5
	<	$\left( \begin{array}{c} 1 \\ 2 \end{array} \right. \left. \begin{array}{c} 1 \\ 1 \end{array} \right)$	7.1.1, 7.5

	<		$\approx \{0, 2\}$	$\begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 3 & 2 \end{pmatrix}$	7.2.2, 7.9 7.7.3, 7.9
	<			$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	7.9
	<			$\begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.9
	<			$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.7.3, 7.9
	<			$\begin{pmatrix} 3 & 1 \\ 3 & 2 \\ 4 & 1 \\ 4 & 5 \\ 5 & 2 \end{pmatrix}$	7.9
	<			$\begin{pmatrix} 3 & 1 \\ 3 & 4 \\ 4 & 2 \end{pmatrix}$	7.9
	<			$\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$	7.6.2, 7.9
	<		Smallest clone (Theorem 1)		7.5
	<		Smallest clone (Theorem 1)		7.5
	<		Smallest clone (Theorem 1)		7.5
	<		Smallest clone (Theorem 1)		7.6.2, 7.9
	<		Smallest clone (Theorem 1)		7.6.2, 7.9
	<		Smallest clone (Theorem 1)		7.6.2, 7.9
	<		Smallest clone (Theorem 1)		7.6.2, 7.9
	<		Smallest clone (Theorem 1)		7.6.2, 7.9
	<		Smallest clone (Theorem 1)		7.6.2, 7.9
	<		Smallest clone (Theorem 1)		7.5
	<		Smallest clone (Theorem 1)		7.5
	<		Smallest clone (Theorem 1)		7.5
	<		Smallest clone (Theorem 1)		7.10
	<		Smallest clone (Theorem 1)		7.1, 7.3, 7.4, 7.6, 7.7, 7.8, 7.9
	<		Smallest clone (Theorem 1)		7.1, 7.3, 7.4, 7.6, 7.7, 7.8, 7.9
	<		Smallest clone (Theorem 1)		7.1, 7.3, 7.4, 7.6, 7.7, 7.8, 7.9
	<		Smallest clone (Theorem 1)		7.7.3, 7.9

- <  Smallest clone (Theorem 1) 7.7.3, 7.9
- <  Smallest clone (Theorem 1) 7.7.3, 7.9
- <  Smallest clone (Theorem 1) 7.2, 7.6, 7.7.2, 7.7.3, 7.8, 7.10
- <  Smallest clone (Theorem 1) 7.2, 7.6, 7.7.2, 7.7.3, 7.8, 7.10
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- <  Smallest clone (Theorem 1) 7.9, 7.10

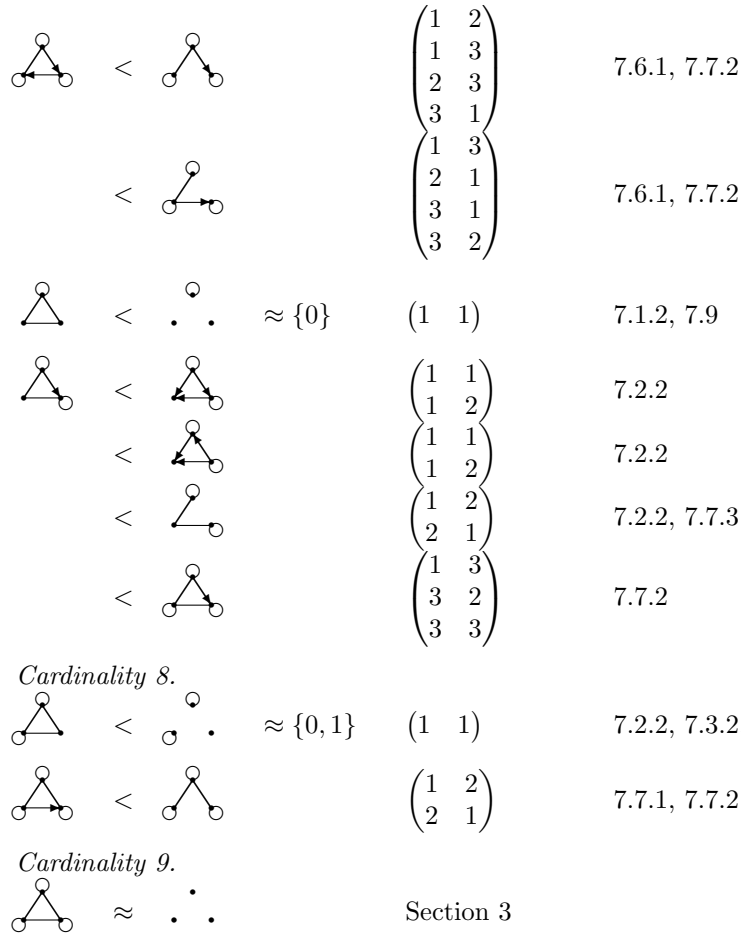
*Cardinality 5.*

-  <   $\approx \{0, 1\}$   $\begin{pmatrix} 1 & 1 \end{pmatrix}$  7.2, 7.3, 7.4
-  <   $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  7.1.1
-  <   $\approx \{0\}$   $\begin{pmatrix} 1 & 1 \end{pmatrix}$  7.1.2, 7.9
-  <   $\begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 3 \end{pmatrix}$  7.1.1, 7.3.2
- <   $\begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$  7.1.1, 7.3.2, 7.7.1
- <   $\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$  7.3.2
-  <   $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$  7.2.1, 7.3.1, 7.4.1
- <   $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  7.2.1, 7.3.1, 7.4.1, 7.6.1
- <   $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$  7.2.1, 7.3.1, 7.4.1, 7.7.2
- <   $\begin{pmatrix} 1 & 3 \\ 2 & 3 \\ 3 & 2 \end{pmatrix}$  7.6.1, 7.7.2
-  <   $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$  7.2.1, 7.3.1, 7.4.1
-  <   $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$  7.2.1, 7.3.1, 7.4.1
- <   $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$  7.2.1, 7.3.1, 7.4.1
-  <   $\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$  7.7.3, 7.9
-  <   $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  7.1.2, 7.9

	$<$		$\begin{pmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 3 \\ 2 & 5 \\ 4 & 5 \end{pmatrix}$	7.1.2, 7.9
	$<$		$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.7.3, 7.9
	$<$		$\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}$	7.1.2, 7.9
	$<$	$\cdot \cdot \cdot$	Maximal clone	7.6, 7.9
	$<$		$\begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 3 & 3 \end{pmatrix}$	7.1.1, 7.7.1
	$<$		$\begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.2.1
	$<$		$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	7.2.1, 7.6.1
	$<$		$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	7.2.2, 7.7.3
	$<$		$\begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 3 & 2 \end{pmatrix}$	7.2.1
	$<$		$\begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}$	7.2.1
	$<$		$\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 1 \end{pmatrix}$	7.2.1
	$<$		$\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 1 \\ 3 & 4 \end{pmatrix}$	7.6.1, 7.7.2
	$<$		Theorem 3	7.6.2
	$<$		Theorem 3	7.2, 7.6, 7.7.2, 7.7.3, 7.8
	$<$		Theorem 3	7.7.3
	$<$		Theorem 3	7.7.3
	$<$		Theorem 3	7.7.2, 7.8
	$<$		Theorem 3	7.7.2, 7.8

	<		Theorem 3	7.2.2
	<		Theorem 3	7.2.1, 7.6.1, 7.7.2, 7.7.3
	<		Theorem 3	7.2.1, 7.6.1, 7.7.2, 7.7.3
	<		Theorem 3	7.2.1, 7.6.1
	<		Theorem 3	7.2.1, 7.6.1
	<		Theorem 3	7.10
	<		Theorem 3	7.2.1
	<		Theorem 3	7.2.1
	<		Theorem 3	7.6.1, 7.7.2
	<		Theorem 3	7.2.2, 7.7.2, 7.7.3
	<		$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.8
	<		$\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}$	7.7.2
	$\approx$		$\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 1 \\ 3 & 4 \\ 4 & 2 \end{pmatrix}$	
	$\approx$		Theorem 2	
<i>Cardinality 6.</i>				
	$\approx$		$\approx \{0, 1\}$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$ and $(1)$
	<		$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	7.2.1, 7.3.2, 7.4.2
	<		$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	7.2.2, 7.3.2, 7.4.2
	<		$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	7.2.2, 7.3.2, 7.4.2
	<		$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.2.2, 7.3.2, 7.4.2
	<		$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.6.1
	<		$\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}$	7.7.2

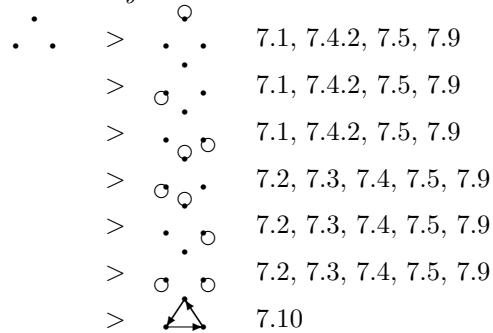
	<		$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	7.1.2
	<		$\begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.1.1
	<		$\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{pmatrix}$	7.1.1, 7.3.2, 7.7.1
	<		$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	7.1.1, 7.7.1
	<		$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.7.1
	<		$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	7.2.1, 7.3.1, 7.4.1
	<		$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	7.2.1, 7.3.1, 7.4.1, 7.6.1
	<		$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$	7.2.1, 7.3.1, 7.4.1
	<		$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$	7.6.1
	<		$\begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 3 \end{pmatrix}$	7.2.1, 7.6.1, 7.7.2
	<		$\begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 1 \\ 3 & 2 \end{pmatrix}$	7.2.1, 7.6.1, 7.7.2, 7.7.3
	<	$\cdot \cdot \cdot$	Maximal clone	7.8
	<		$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	7.2.2, 7.7.3, 7.9
	<	$\cdot \cdot \cdot$		7.10
	<	$\cdot \cdot \cdot$		7.9, 7.10
<i>Cardinality 7.</i>				
	<		$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	7.2.1, 7.3.2, 7.4.2
	<		$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	7.6.1, 7.9
	<	$\cdot \cdot \cdot$	Maximal clone	7.7, 7.9

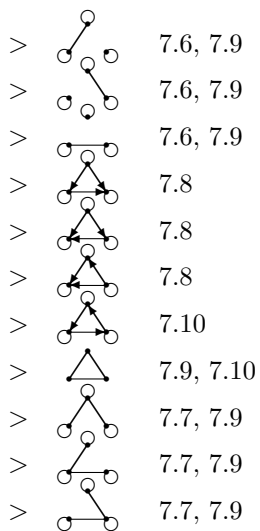
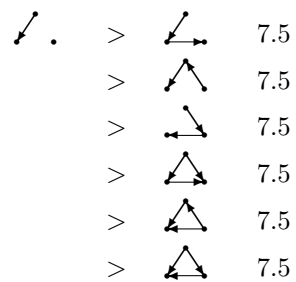
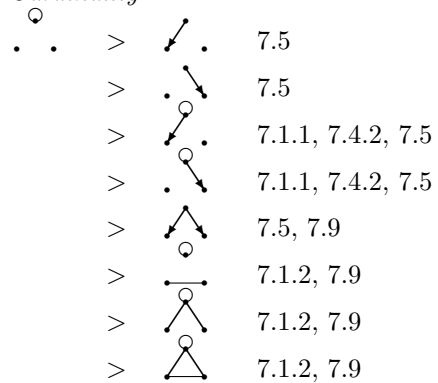
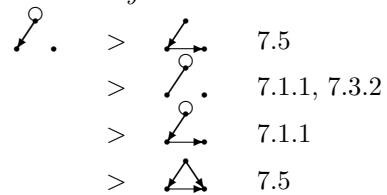































## 6. INVERSE COVERINGS

In this section, we indicate which relations  $\tau$  are such that  $\text{Pol } \rho \supseteq \text{Pol } \tau$  minimally (within the scope of this study) and in which diagrams this covering can be found. The information can be derived from Section 5, but I have found it sufficiently useful in my research to include here.







*Cardinality 0.*

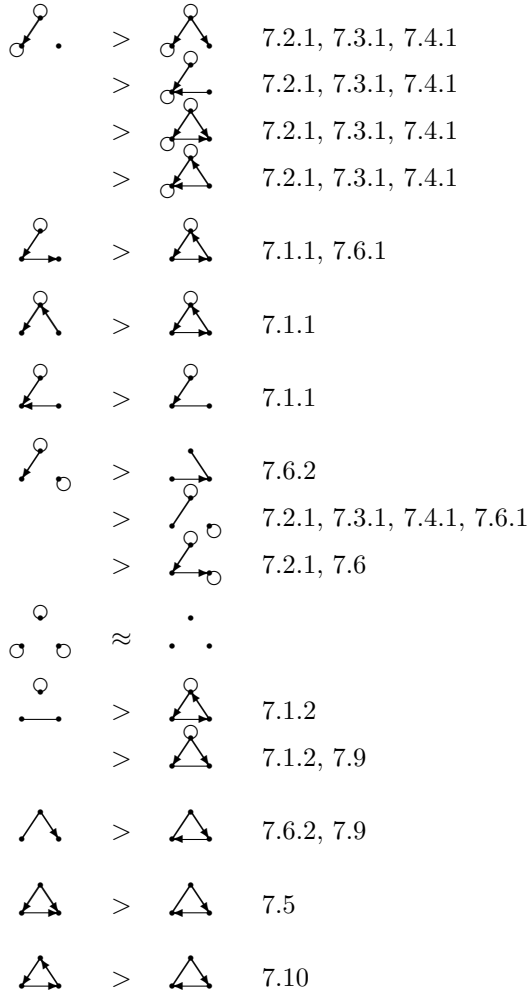
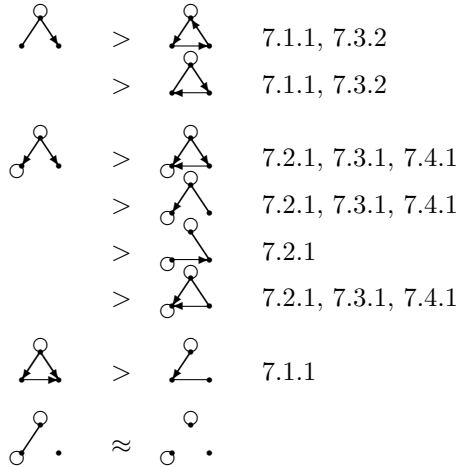


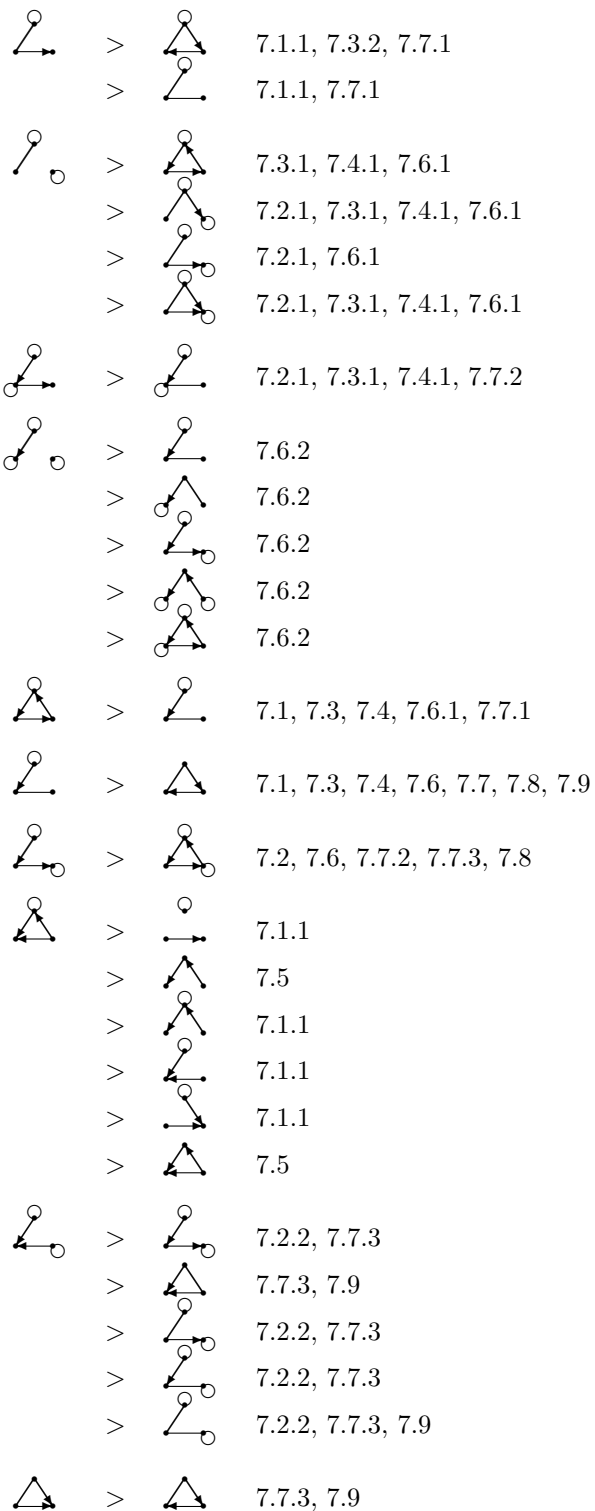
*Cardinality 1.**Cardinality 2.*

- 
  - >  7.1.1
  - >  7.1.1, 7.5
- 
  - >  7.3.2, 7.4.2, 7.5
  - >  7.3.2, 7.4.2, 7.5
  - >  7.9
  - >  7.5, 7.9
  - >  7.2.1, 7.3.1, 7.4.1
  - >  7.2.1, 7.3.1, 7.4.1
  - >  7.2.1, 7.3.1, 7.4.1
  - >  7.2.2, 7.9
  - >  7.2, 7.3, 7.4
  - >  7.2, 7.3, 7.4
  - >  7.2.2, 7.3.2, 7.4.2, 7.9
- 
  - >  7.1.1
- 
  - >  7.5
  - >  7.5
  - >  7.5
  - >  7.5
  - >  7.9
- 
  - >  7.9
  - >  7.9
  - >  7.9
- 
  - >  7.5

*Cardinality 3.*

- 
  - ≈ 
- 
  - >  7.1.1, 7.3.2
  - >  7.1.1, 7.3.2
  - >  7.1.1

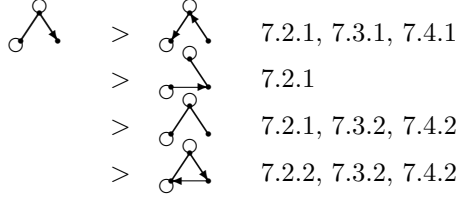
*Cardinality 4.*





7.6.2, 7.9

7.6.2, 7.9

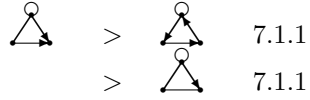
*Cardinality 5.*

7.2.1, 7.3.1, 7.4.1

7.2.1

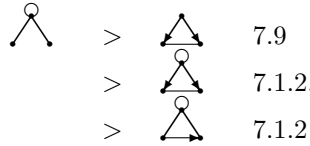
7.2.1, 7.3.2, 7.4.2

7.2.2, 7.3.2, 7.4.2



7.1.1

7.1.1



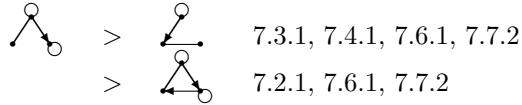
7.9

7.1.2, 7.9

7.1.2

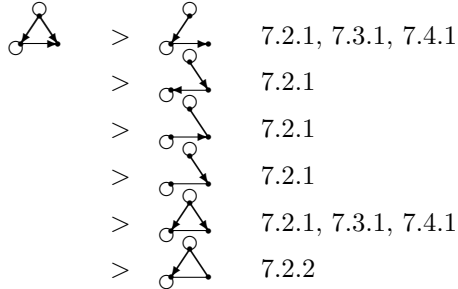


7.1.1, 7.3.2, 7.7.1



7.3.1, 7.4.1, 7.6.1, 7.7.2

7.2.1, 7.6.1, 7.7.2



7.2.1, 7.3.1, 7.4.1

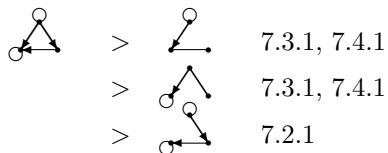
7.2.1

7.2.1

7.2.1

7.2.1, 7.3.1, 7.4.1

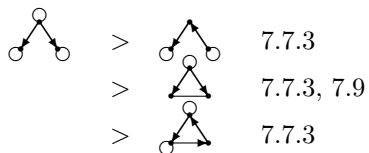
7.2.2



7.3.1, 7.4.1

7.3.1, 7.4.1




































7.2.1

















7.7.3

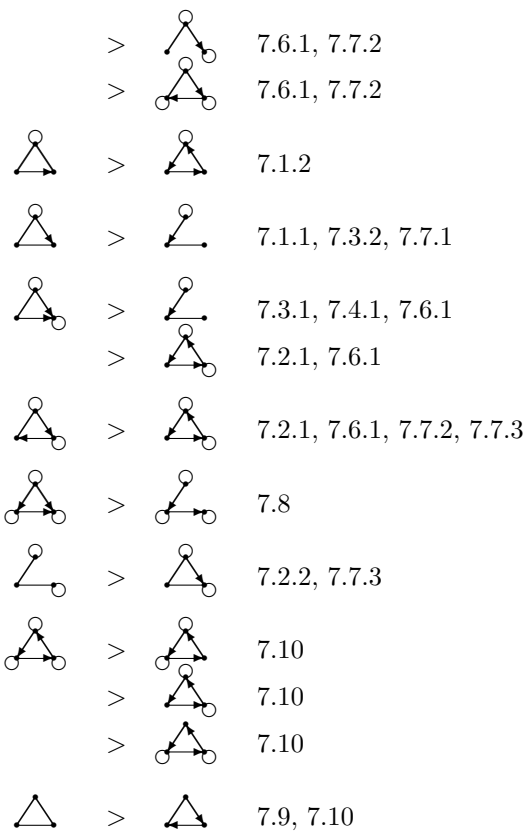
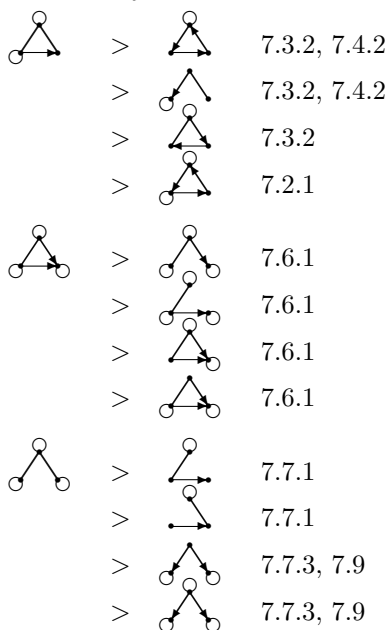
7.7.3, 7.9

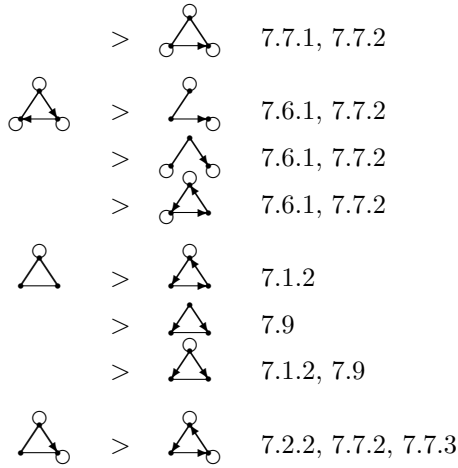
7.7.3

-   $>$   7.7.3
-   $>$   7.1.2, 7.7.3, 7.9
-   $>$   7.6.1
-   $>$   7.6.1
-   $>$   7.6
-   $>$   7.6
-   $>$   7.6.2
-   $>$   7.6.2, 7.9
-   $>$   7.6.1, 7.9
-   $>$   7.1.1, 7.7.1
-   $>$   7.2.1, 7.6.1, 7.7.2, 7.7.3
-   $>$   7.2, 7.6, 7.7.2, 7.7.3, 7.8, 7.10
-   $>$   7.7.2, 7.8
-   $>$   7.7.2, 7.8
-   $>$   7.7.2, 7.8
-   $>$   7.7.2, 7.8
-   $\approx$  
-   $\approx$  

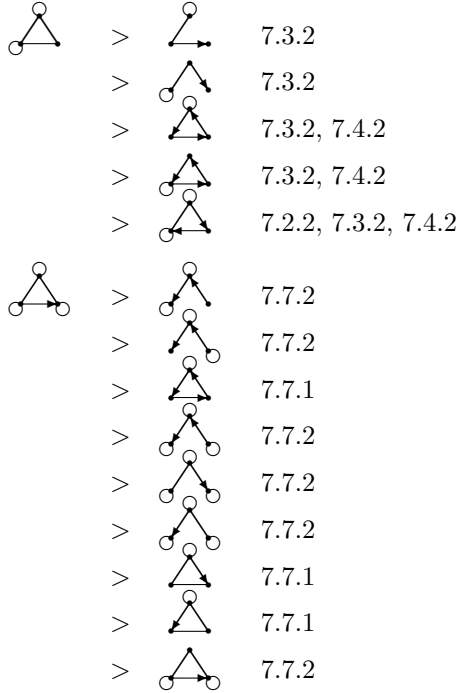
*Cardinality 6.*

-   $\approx$  
-   $>$   7.3.2
-   $>$   7.2.1, 7.3.2, 7.4.2
-   $>$   7.3.2, 7.4.2
-   $>$   7.3.2, 7.4.2
-   $>$   7.2.2
-   $>$   7.6.1, 7.7.2

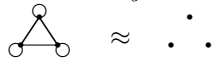
*Cardinality 7.*



*Cardinality 8.*



*Cardinality 9.*



















## 7. INCLUSION DIAGRAM

The inclusion diagram will be separated into 10 sub-diagrams to aid in reading it. These sub-diagrams are either one or more intervals, or a set of clones satisfying a certain property. As in the rest of this paper, we do not repeat isomorphic parts

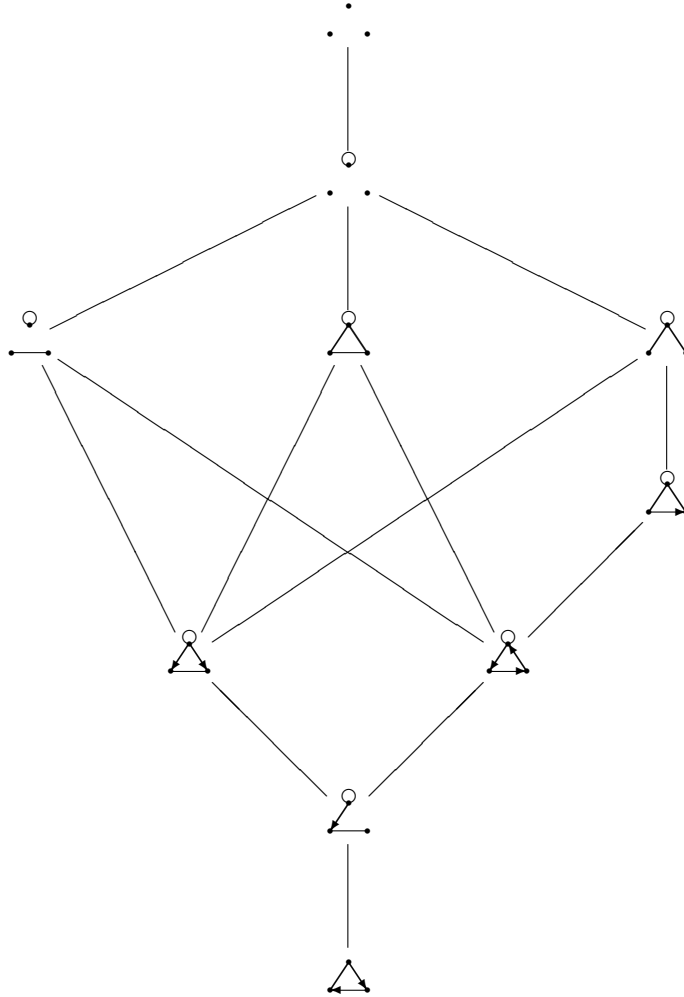
of the inclusion diagram needlessly, but at least one inclusion per type is drawn. Some inclusions are repeated in more than one sub-diagram. Certain sub-diagrams are further subdivided into parts.

All the sub-diagram are Hasse diagrams. That is  $A \subseteq B$  is indicated by linking  $A$  and  $B$  by a line such that  $A$  is lower than  $B$  on the page. Throughout, we write simply the relation to represent the clone of functions preserving that relation.

The sub-diagrams are:

- (A) The interval from  to ; that is, the clones containing the constant function  $c_0$  but no other constant function.
- (B) The interval from  to ; that is, the clones containing the constant functions  $c_0$  and  $c_1$  but not the constant function  $c_2$ .
- (C) The intervals from  and  to  .. 
- (D) The intervals from  to  .. and to 
- (E) The intervals from  and  to the greatest clone.
- (F) The interval from the smallest clone to 
- (G) The interval from the smallest clone to 
- (H) The interval from the smallest clone to ; that is, the clones containing only monotone functions where the ordering is  $2 < 0 < 1$ .
- (I) The clones containing the permutation  $(1, 2)$ .
- (J) The clones containing the permutation  $(0, 1, 2)$ .

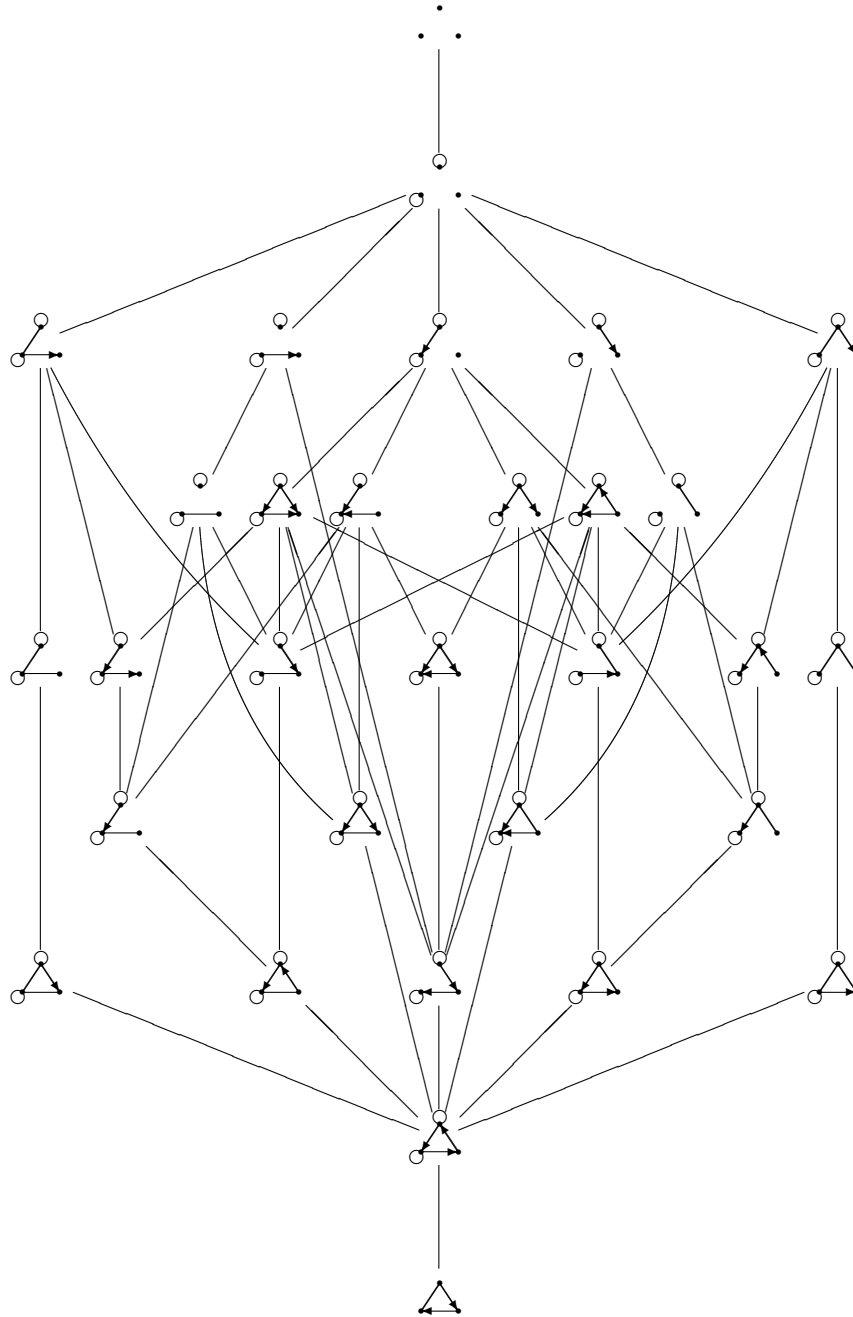


**7.1.2.** *Second part*

7.2. The interval from  $\begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \circ \end{array}$  to  $\begin{array}{c} \circ \\ \circ \end{array}$  ..

They are the clones containing the constant functions  $c_0$  and  $c_1$  but not the constant function  $c_2$ . The greatest and smallest clones are also indicated.

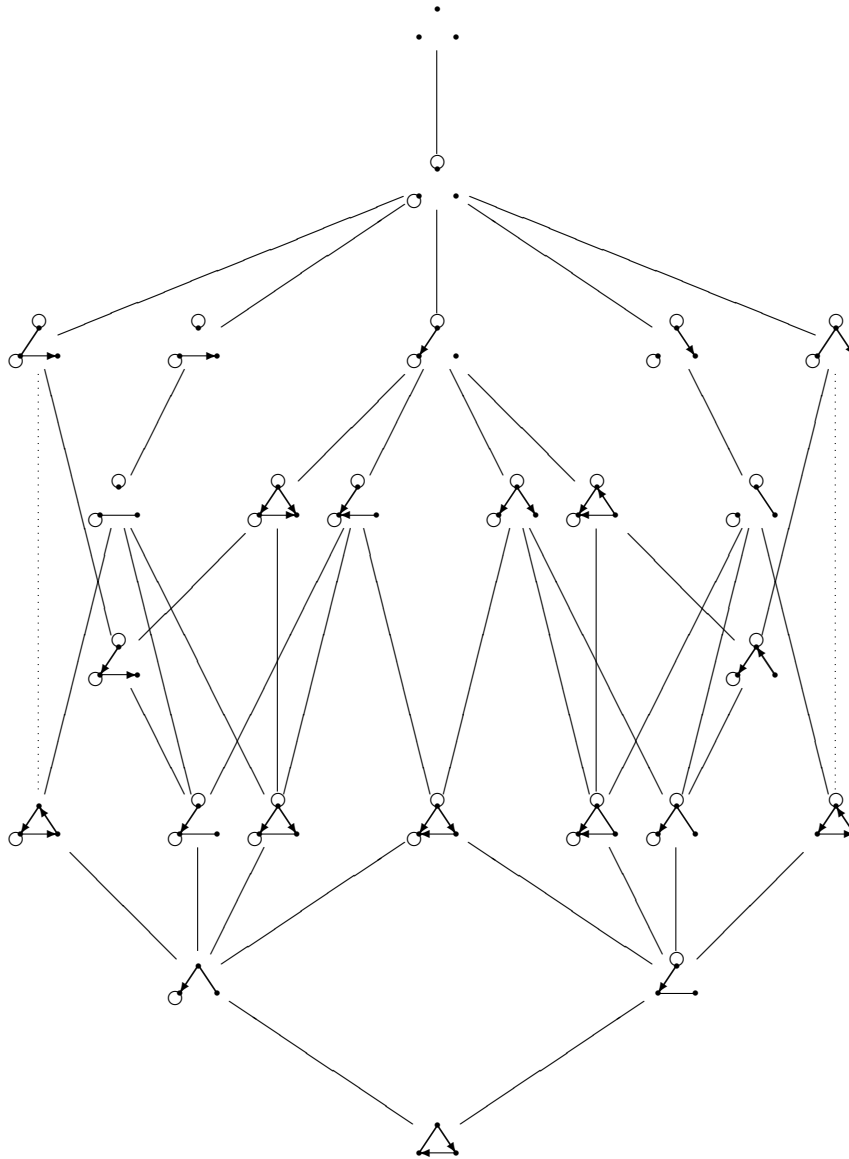
7.2.1. *First part*



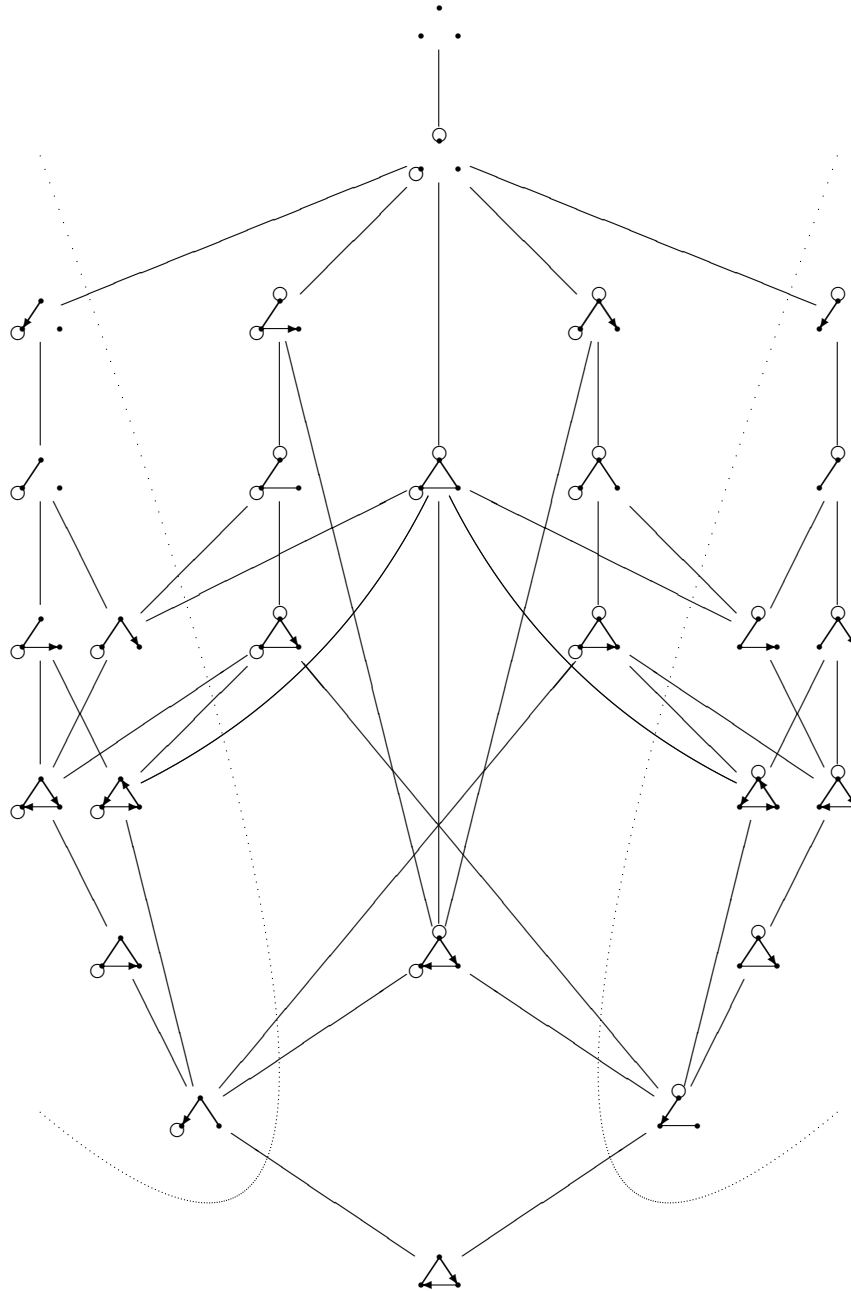


**7.3.** The intervals from  $\begin{array}{c} \circ \\ \swarrow \rightarrow \end{array}$  to  $\begin{array}{c} \circ \\ \circ \end{array}$  .., and from  $\begin{array}{c} \circ \\ \swarrow \searrow \end{array}$  to  $\begin{array}{c} \circ \\ \circ \end{array}$  ..  
 The greatest and smallest clones are also indicated.

**7.3.1.** *First part.* A dotted line means that not all clones are shown in the inclusion; details in Diagram 7.3.2.

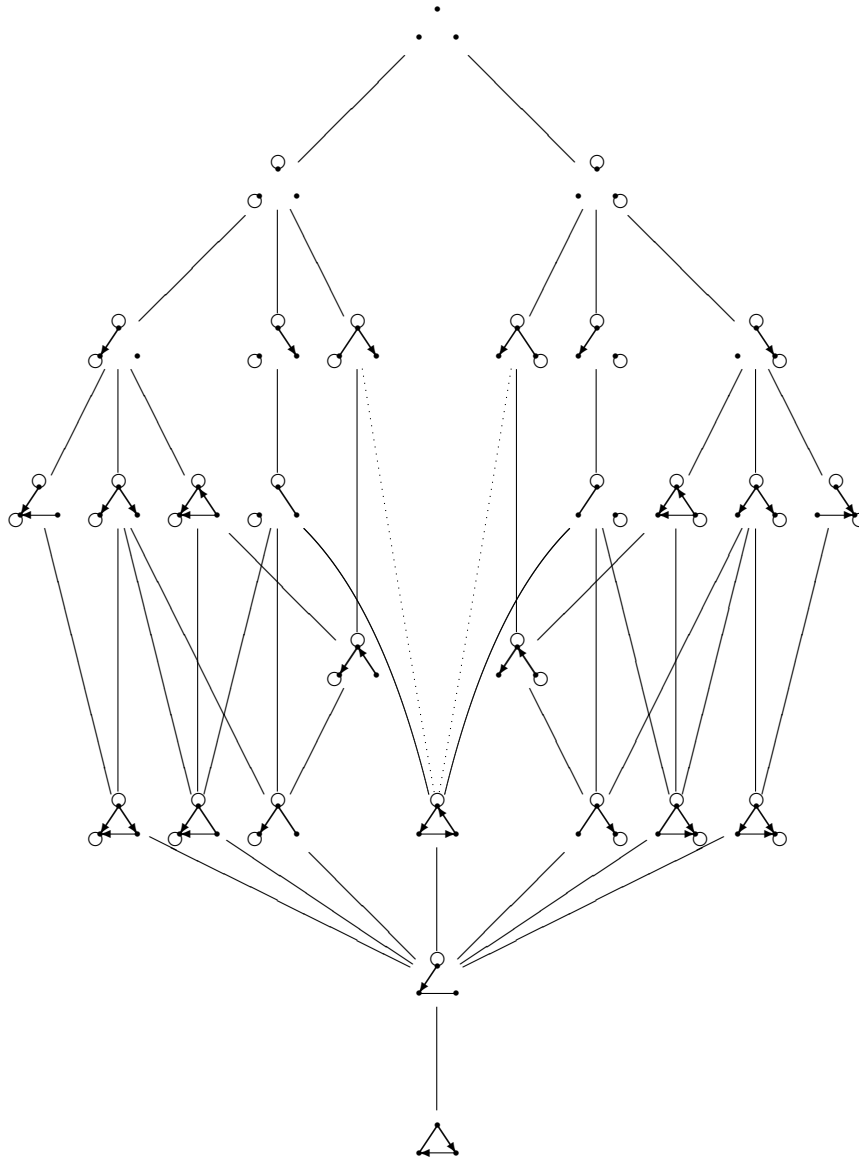


**7.3.2. Second part.** The intervals from  $\begin{array}{c} \circ \\ \swarrow \rightarrow \\ \cdot \end{array}$  to  $\begin{array}{c} \circ \\ \swarrow \nearrow \\ \cdot \end{array}$  and from  $\begin{array}{c} \circ \\ \swarrow \nwarrow \\ \cdot \end{array}$  to  $\begin{array}{c} \circ \\ \swarrow \rightarrow \\ \cdot \end{array}$  are partially shown; the first is in Diagram 7.1.1.

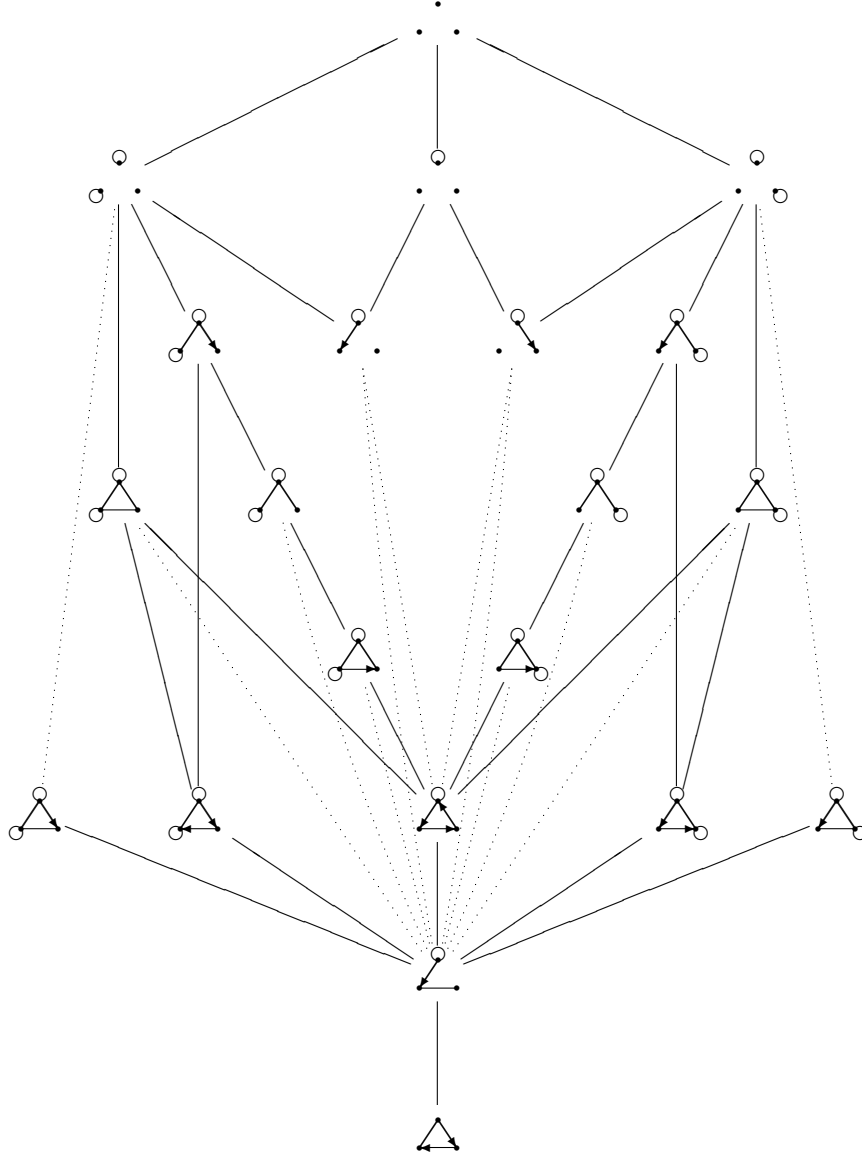


7.4. The intervals from  $\begin{array}{c} \circ \\ \swarrow \searrow \\ \circ \end{array}$  to  $\begin{array}{c} \circ \\ \circ \end{array}$ , and from  $\begin{array}{c} \circ \\ \swarrow \searrow \\ \circ \end{array}$  to  $\begin{array}{c} \circ \\ \circ \end{array}$ .  
 The greatest and smallest clones are also indicated.

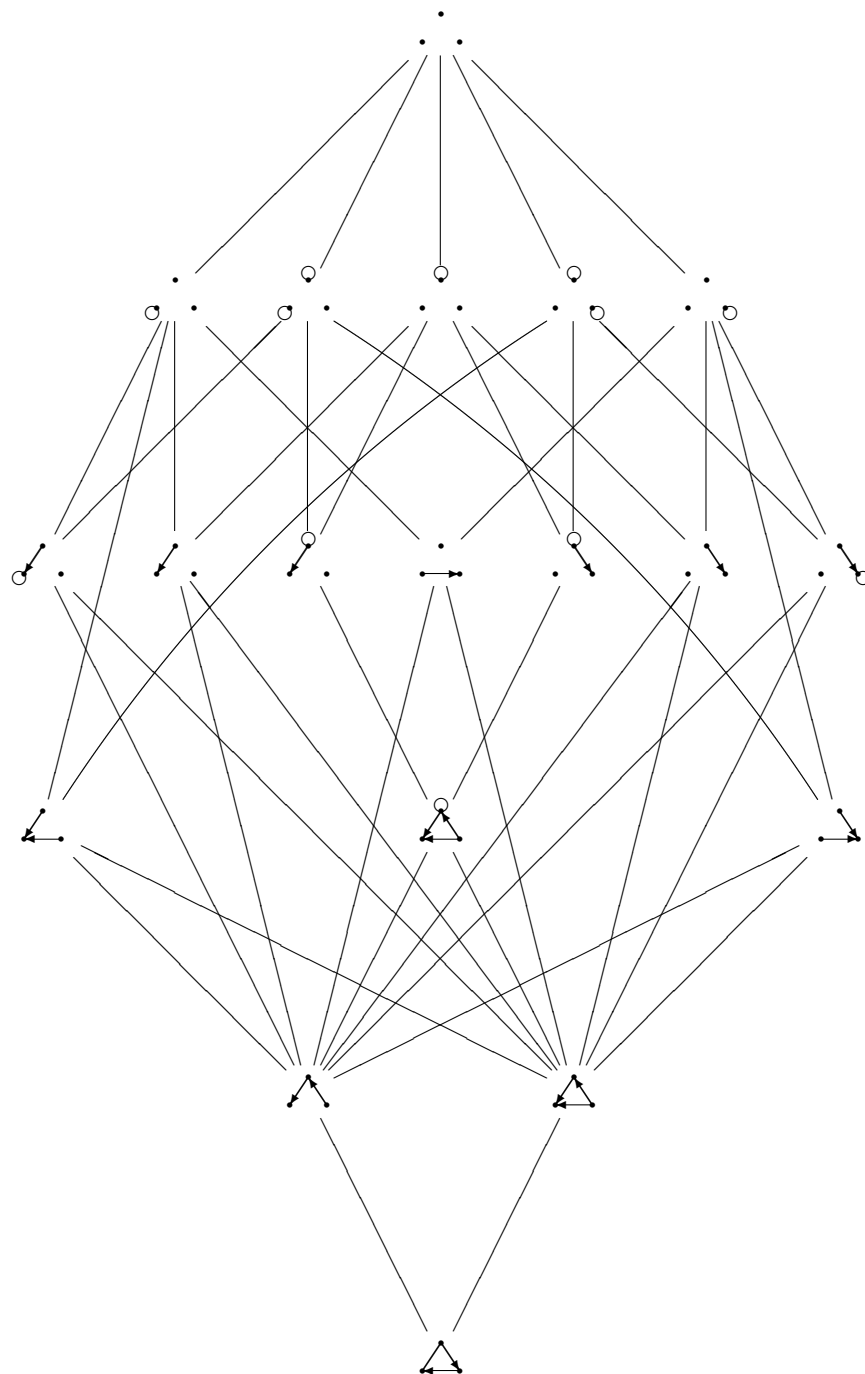
7.4.1. *First part.* A dotted line means that not all clones are shown in the inclusion; details in Diagram 7.3.2.




**7.4.2. Second part.** A dotted line means that not all clones are shown in the inclusion; details in Diagrams 7.3.2 and 7.1.1. The clone represented by the relation  $\cdot \cdot$  is also shown.

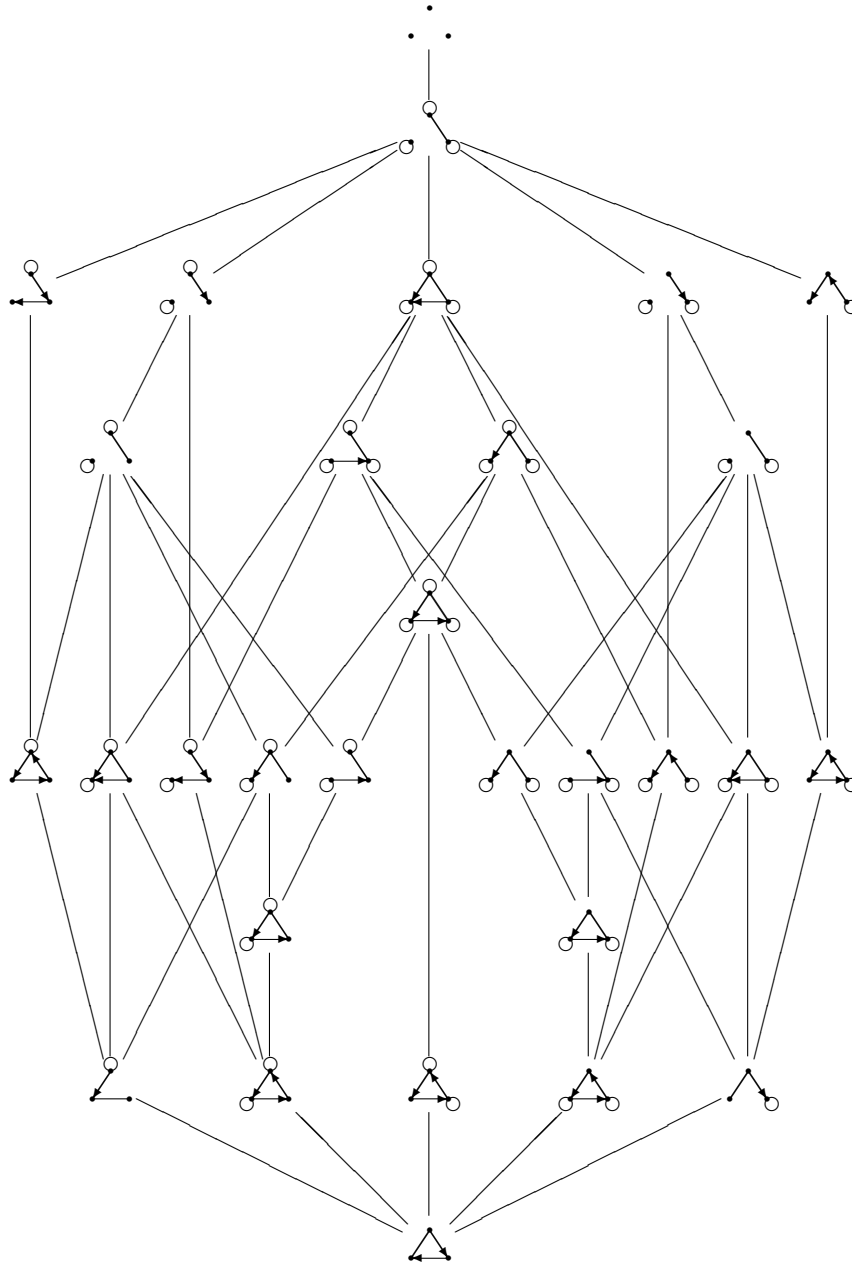


7.5. The intervals from  $\rightarrow$  and  $\triangleleft$  to the greatest clone.  
 The smallest clone is also indicated.

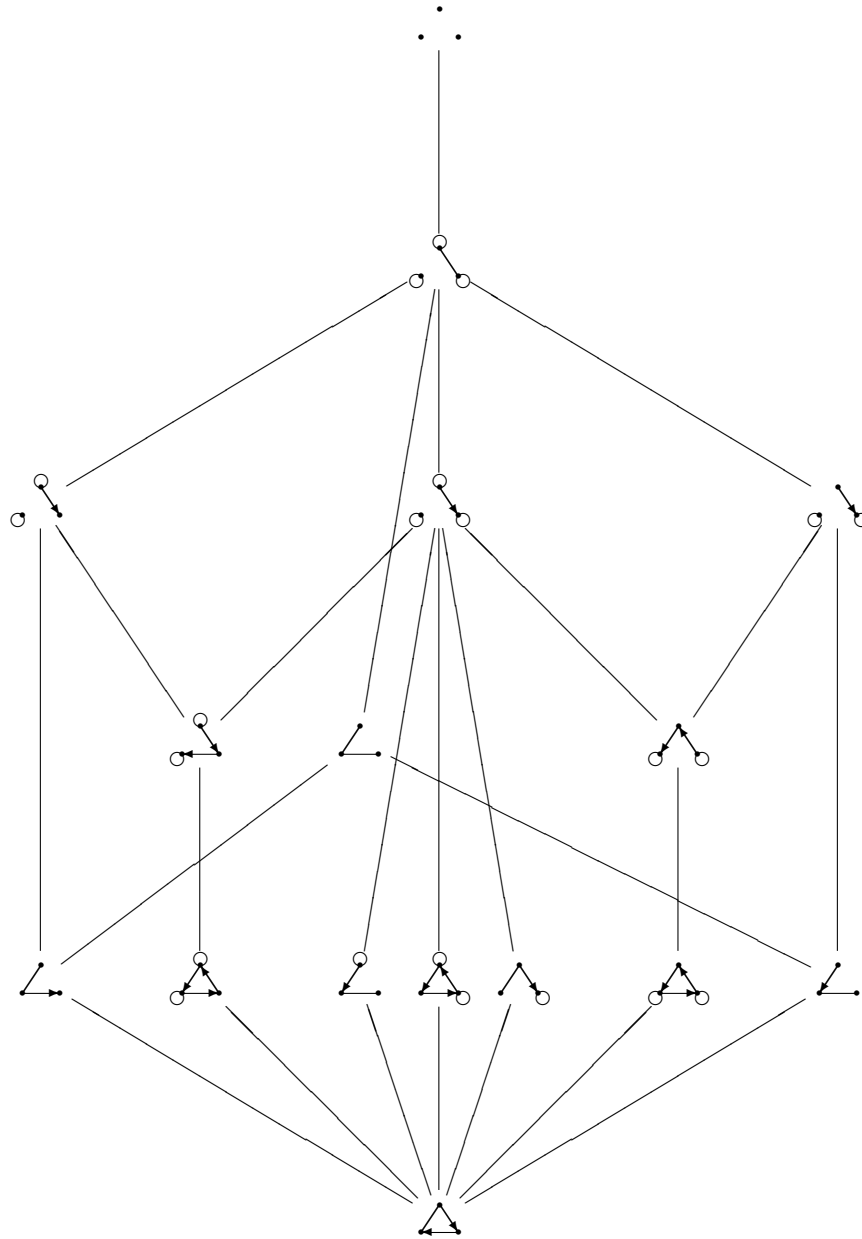



7.6. The intervals from the smallest clone to   
 The greatest clone is also indicated.

7.6.1. *First part*

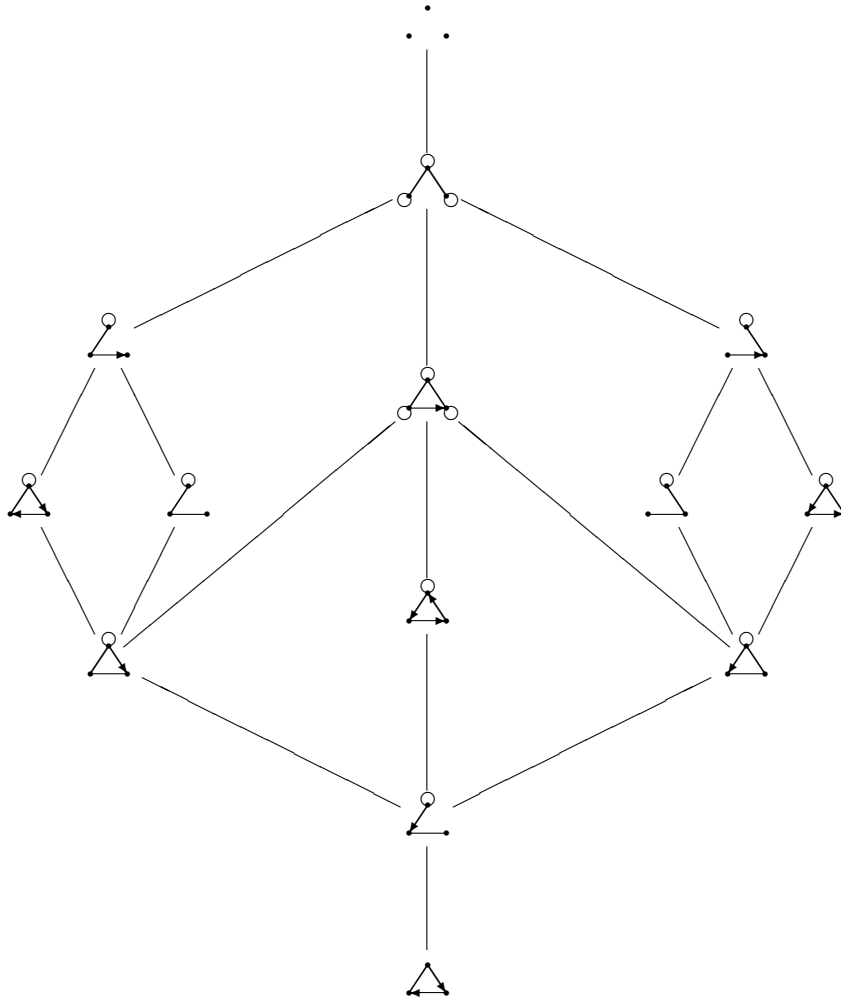


7.6.2. *Second part*

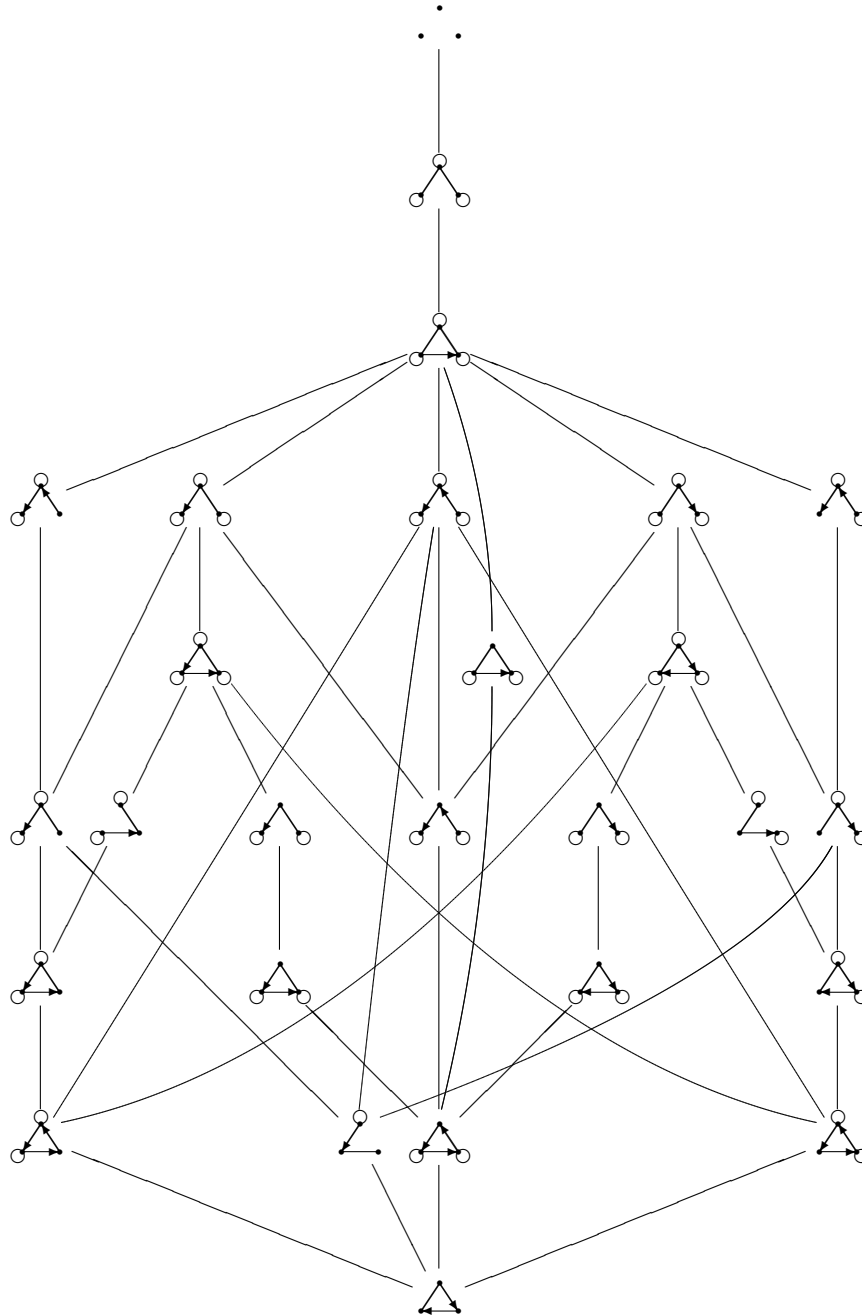


7.7. The interval from the smallest clone to   
The greatest clone is also indicated.


7.7.1. *First part*

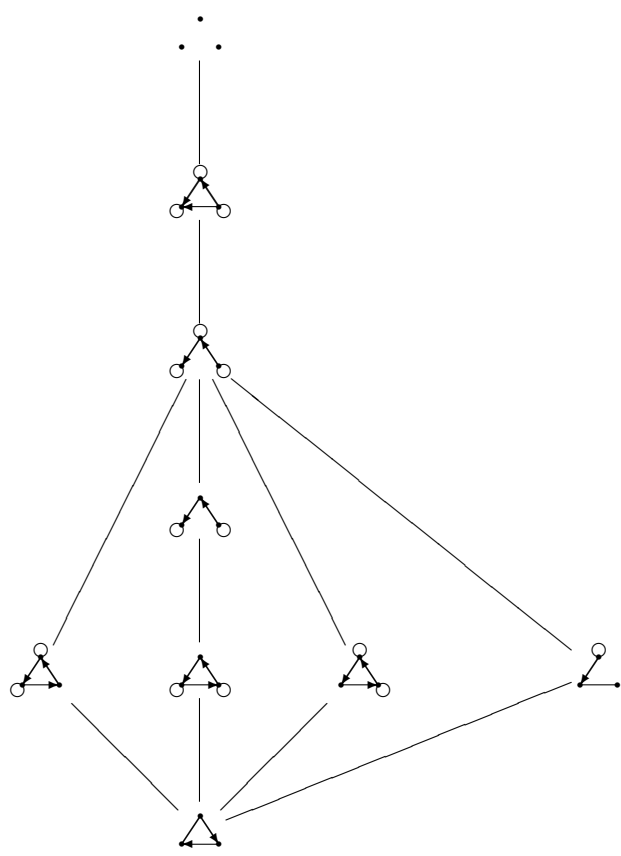


7.7.2. *Second part*



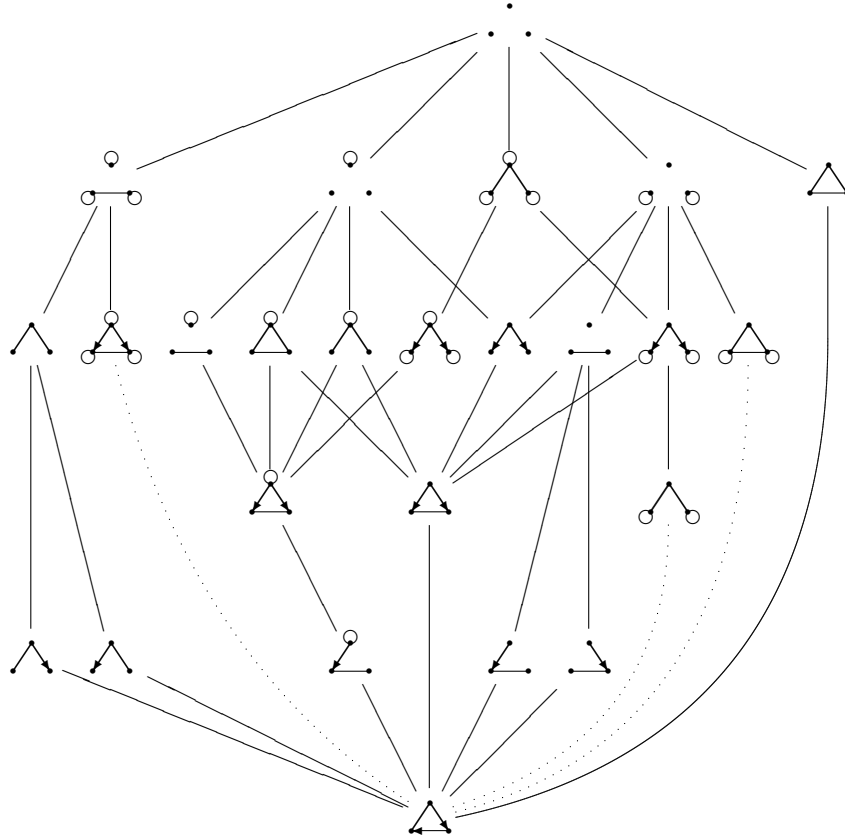


7.8. The interval from the smallest clone to  They are the clones containing only monotone functions where the ordering is  $2 < 0 < 1$ . The greatest clone is also indicated.



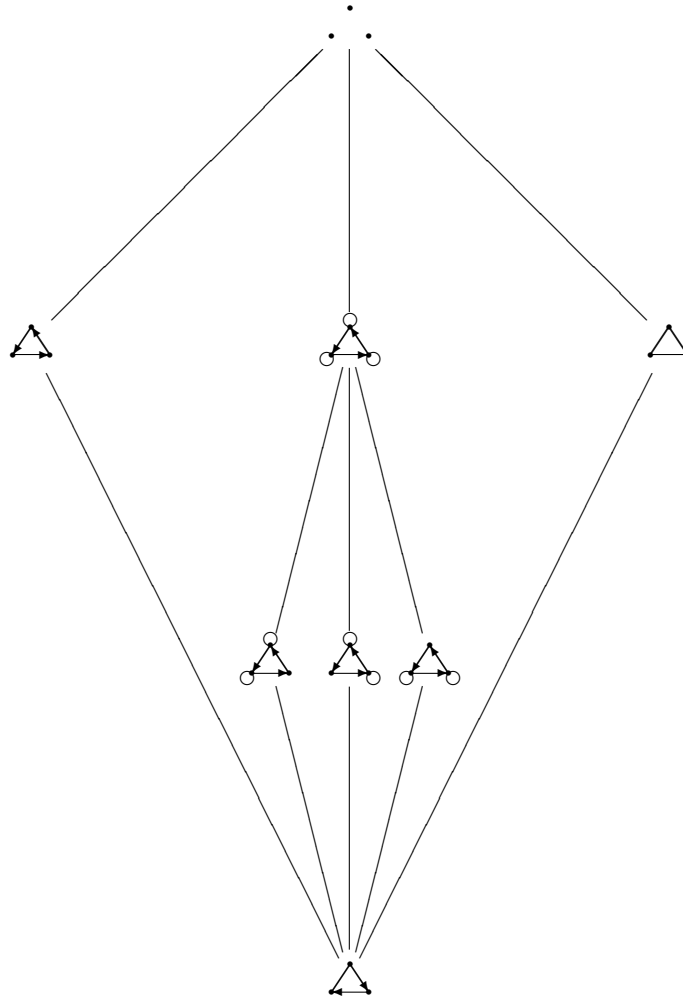
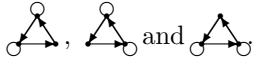
**7.9.** The clones containing the transposition  $(1, 2)$ .

Are also shown the smallest clone and the clones represented by the relations  $\triangleleft$ ,  $\triangleright$ ,  $\triangleleft$ ,  $\triangleright$  and  $\triangleleft$ . A dotted line signifies that not all the inclusions are indicated; details can be found in Diagram 7.6.1 and Diagram 7.2.2.



**7.10.** The clones containing the permutation  $(0, 1, 2)$ .

Are also shown: the smallest clone and the clones represented by the relations



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ANNE FEARNLEY, DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE, UNIVERSITÉ DE MONTRÉAL, MONTREAL, QC, CANADA

*E-mail address:* `fearnley@dms.umontreal.ca`

*URL:* `http://www.dms.umontreal.ca/~fearnley`