POST'S LATTICE

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Abstract. A drawing of the lattice of clones on 2 elements using Post's notation. A description of the clones and several other useful information about clones on 2 elements.

1. Introduction

The purpose of this paper is not to prove Post's classification of Boolean clones [1]. It is rather an exposition of his results along with many of the results about clones on 2 elements contained in Ágnes Szendrei's Clones in Universal Algebra [3] and some things I found myself. The results in this paper are the ones I have found useful in my own research.

The article is made up of the drawing of the lattice (mostly taken from [3]), a description of the clones including generators and some relations, a table to aid in finding which clone you are dealing with, and the monoidal intervals in the lattice of clones on 2 elements. I have included the original Latex document as well as the more usual PDF so that mathematicians may use parts of the drawing of the lattice in their own papers.

2. Definitions

Let A be a finite set and n a positive integer. An *n-ary operation* on A is a function $f : A^n \to A$. The set of all n-ary operations on A is denoted by $\mathcal{O}_A^{(n)}$, and $\mathcal{O}_A := \bigcup_{0<n<\omega} \mathcal{O}_A^{(n)}$. For $1 \leq i \leq n$, the n-ary *i-th projection* is defined as $e_i^{(n)}(x_1, \dots, x_n) = x_i$ for all $x_1, \dots, x_n$. We write e for the identity operation. For $a \in A$, the n-ary *constant operation* a is defined as $c_a^{(n)}(x_1, \dots, x_n) = a$ for all $x_1, \dots, x_n$. We write simply c_a for the unary constant operations $c_a^{(1)}$.

For $f \in \mathcal{O}^{(n)}$, and $g_1, \dots, g_n \in \mathcal{O}^{(m)}$, we define their composition to be the m-ary operation $f[g_1, \dots, g_n]$ defined by

$$f[g_1, \dots, g_n](x_1, \dots, x_m) = f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m))$$

A *clone* on A is a subset F of $\mathcal{O}_A$ which contains all projections and is closed under composition. It is well known and easy to prove that the intersection of an arbitrary set of clones on A is a clone on A. Thus for $F \subseteq \mathcal{O}_A$, there exists the least clone containing F, called the clone *generated* by F and denoted by $\langle F \rangle$. The clones on A, ordered by inclusion, form a complete lattice, $\mathcal{L}_A$.

Date: October 18, 2006.

1991 *Mathematics Subject Classification.* Primary: 08A40; Secondary: 03B05.

Key words and phrases. clones, propositional logic.

Let h be a positive integer. An *h-ary relation* ρ is a subset of A^h. Let $f \in \mathcal{O}^{(n)}$, and let ρ be an h-ary relation on A. The operation f *preserves* ρ if for all $(a_{1,i}, a_{2,i}, \ldots, a_{h,i}) \in \rho$ $(i = 1, \ldots, n)$,

$$(f(a_{1,1}, a_{1,2}, \ldots, a_{1,n}), f(a_{2,1}, a_{2,2}, \ldots, a_{2,n}), \ldots, f(a_{h,1}, a_{h,2}, \ldots, a_{h,n})) \in \rho$$

The set of operations on A preserving ρ is a clone denoted by $\operatorname{Pol}\rho$.

Consider a transformation monoid M of unary operations on A; i.e. M contains the identity self-map e and is closed under the usual composition. Denote by $\operatorname{Int}(M)$ the set of clones C on A such that the unary operations of C are exacty M. It is well known that $\operatorname{Int}(M)$ is an interval in the lattice of clones on A, called the *monoidal interval* of M.

3. The lattice of clones on 2 elements

From now on, we will assume that A has two elements. It is most usual to consider $A = \{0, 1\}$, but Post prefered to consider the set $\{+, -\}$ where $+$ corresponds to true (or is a member of the class) and $-$ to false (or is not a member of the class). Throughout, I will consider clones on $\{0, 1\}$.

The lattice of clones on 2 elements can be found in Figure 1. It is drawn in the usual way; if a clone A is linked to a clone B by a line and A is lower than B on the page, then $A \subset B$. There are dotted lines to show inclusion for each of the 8 countably infinite families of clones F_i^n for $i = 1, \ldots, 8$. Note that for each $i \in \{1, \ldots, 8\}$, $F_i^\infty = \bigwedge_{n=2}^{\infty} F_i^n = \bigcap_{n=2}^{\infty} F_i^n$.

The clones are written using Post's notation. You may notice that there seem to be some symbols missing, for example S_1. This is because Post was dealing with sets of operations closed under composition but not necessarily containing all the projections. Since I am only interested in clones, I have left the others out of the diagram. In this and in the placement of the clones on the page, I am following [3].

3.1. Closed sets which are not clones.

- $O_1 = \{e\}$
- $O_2 = \{c_1\}$
- $O_3 = \{c_0\}$
- $O_4 = \{e, \neg\}$
- $O_5 = \{e, c_1\}$
- $O_6 = \{e, c_0\}$
- $O_7 = \{c_0, c_1\}$
- $O_8 = \{e, c_0, c_1\}$
- $O_9 = \{e, c_0, c_1, \neg\}$
- $R_2 = \{c_1^{(n)} \mid n \in \mathbb{N}\}$
- $R_3 = \{c_0^{(n)} \mid n \in \mathbb{N}\}$
- $R_5 = \{e\} \cup \{c_1^{(n)} \mid n \in \mathbb{N}\}$
- $R_7 = \{e\} \cup \{c_0^{(n)} \mid n \in \mathbb{N}\}$
- $R_9 = \{c_0^{(n)}, c_1^{(n)} \mid n \in \mathbb{N}\}$
- $R_{10} = \{e\} \cup \{c_0^{(n)}, c_1^{(n)} \mid n \in \mathbb{N}\}$
- $R_{12} = \{e, \neg\} \cup \{c_0^{(n)}, c_1^{(n)} \mid n \in \mathbb{N}\}$
- $S_1 = \{e\} \cup \{\vee^{(n)} \mid n \in \mathbb{N}\}$
  where $\vee^{(n)}$ is the n-ary $\vee$ defined by $\vee^{(n)}(a_1, \ldots, a_n) = a_1 \vee \ldots \vee a_n$.

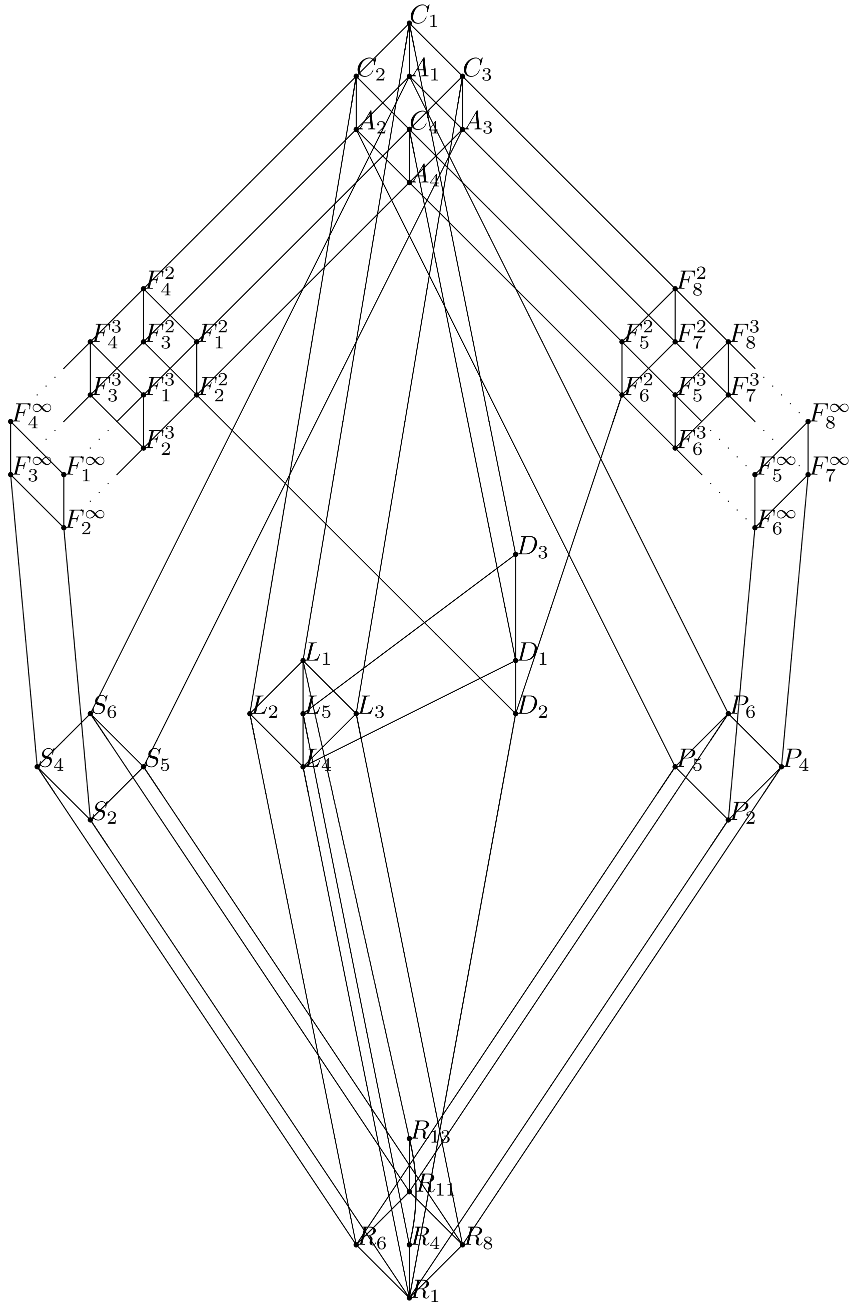


FIGURE 1. The lattice of clones on 2 elements

- $S_3 = \{e\} \cup \{c_1^{(n)}, \vee^{(n)} \mid n \in \mathbb{N}\}$
- $P_1 = \{e\} \cup \{\wedge^{(n)} \mid n \in \mathbb{N}\}$ where $\wedge^{(n)}$ is defined like $\vee^{(n)}$
- $P_3 = \{e\} \cup \{c_0^{(n)}, \wedge^{(n)} \mid n \in \mathbb{N}\}$

4. Description of the clones

In the description tables (Figures 1, 2, 3, 4, 5), each clone is identified using Post's notation. For each of the infinite families, F_i^n for $i = 1, \ldots, 8$, I write F_i^2 seperately because it is a special case, and F_i^n will be understood to begin at $n = 3$.

In a lattice L, $a \in L$ is *join-irreducible* if and only if $a = b \vee c$ implies that $a = b$ or $a = c$, and *meet-irredicible* is defined dually. Any clone can be written as a join of join-irreducible clones or as a meet of meet-irreducible clones. The join-irreducible clones will be indicated by a check, $\check{C}$, and the meet-irredicible clones by a carat, $\hat{C}$.

A clone is often described as the clone generated by certain operations (the 'Generators' column). These generators are either taken directly from [1], or can be easily infered from it. Here is a list of the operations used in the tables.

(1) The constants, c_0, c_1 and the identity e,
(2) The standard logical operators: $\vee, \wedge, \neg, \leftrightarrow$, and operations derived from them written as logical sentences,
(3) Binary and ternary additions mod 2, $+$ and $+^{(3)}$,
(4) The majority operation, maj, defined by $maj(x, y, z) = i$ if $x = y = i$ or $x = z = i$ or $y = z = i$,
(5) The n-ary operations h_n and H_n for all $n = 2, 3, \ldots$, which are defined as $h_n(x_1, \ldots, x_{n+1}) = 0$ if and only if $\sum_{i=1}^{n+1} x_i \leq 1$, and $H_n(x_1, \ldots, x_{n+1}) = 1$ if and only if $\sum_{i=1}^{n+1} x_i \geq n$ (note that $h_2 = H_2 = maj$).

The 'Relation' associated to a clone C is a relation ρ such that $\operatorname{Pol} \rho = C$. There can be many possible relations that have that property; I have usually chosen one with the smallest arity, and after that, the smallest cardinality. The relations are usually written as a set of vertical tuples. In some cases, the relation is the graph of some operation f, and is written as $f^{\square}$. If the relation is not from me, the source is indicated.

The 'Monoid' for a clone is the monoid of unary operations in that clone. It will be discussed in greater detail in Section 5.

The maximal clones (i.e. those that are included in only the largest clone, C_1) are C_2, C_3, A_1, L_1 and D_3. The minimal clones (i.e. those that include only the smallest clone, R_1) are R_4, R_6, R_8, S_2, P_2, L_4, D_2.

5. Monoidal intervals

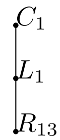
Post named the unary operations on 2 elements using greek letters with subsript 1 to denote that the operations are unary. They are $\alpha_1 = e$, $\beta_1 = c_1$, $\gamma_1 = c_0$, $\delta_1 = \neg$. He notes that for any operation $f : A^n \to A$ (Post says membership function), one may associate a unary operation $d : A \to A$ defined by $d(a) = f(a, \ldots, a)$ for all $a \in A$. Post calls this the *associated first order function* and calls f an α, β, γ or δ-function depending on if its associated first order function is α_1, β_1, γ_1 or δ_1.

In a similar way, for any set of operations closed under composition, one can define the *associated first order system* defined by its associated first order functions. There are 9 possible first order systems, of which 6 contains at least one α-function (corresponding to e). Each closed set of operation must correspond to one of the 9 first order systems. Post uses this fact throughout [1] to organize his classification.

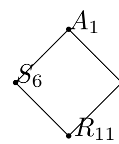
Similarly, each clone must correspond to one of the 6 first order systems containing α_1. This is the same as considering the monoid M of unary operations in a clone C. As we saw in the definitions in Section 2, we then say that C is in the monoidal interval of M. The monoidal intervals partition the whole lattice.

There are 6 monoidal intervals in the lattice of clones on 2 elements. Three are finite and three are infinite.

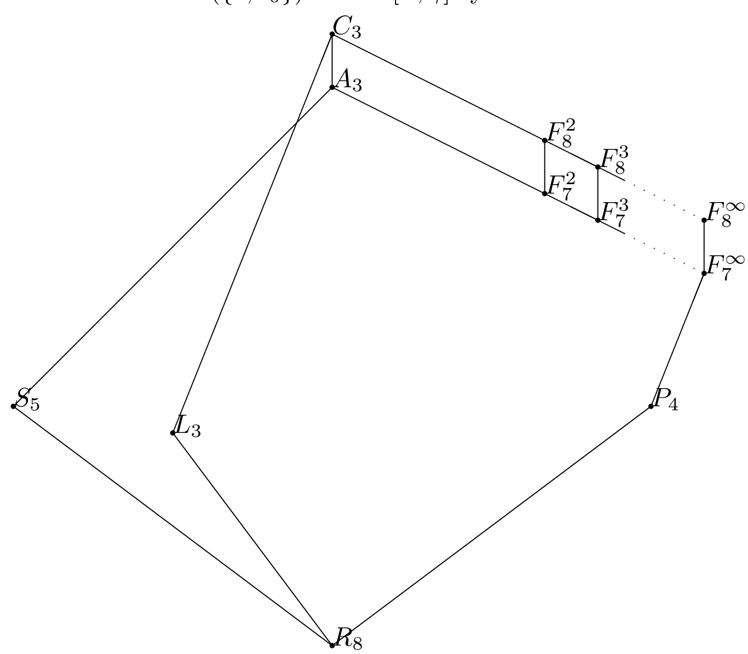
5.1. Monoidal interval $\mathrm{Int}(\{e, c_0, c_1, \neg\})$ or the $[\alpha, \beta, \gamma, \delta]$-systems.



5.2. Monoidal interval $\mathrm{Int}(\{e, c_0, c_1\})$ or the $[\alpha, \beta, \gamma]$-systems.

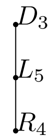


5.3. Monoidal interval $\mathrm{Int}(\{e, c_0\})$ or the $[\alpha, \gamma]$-systems.

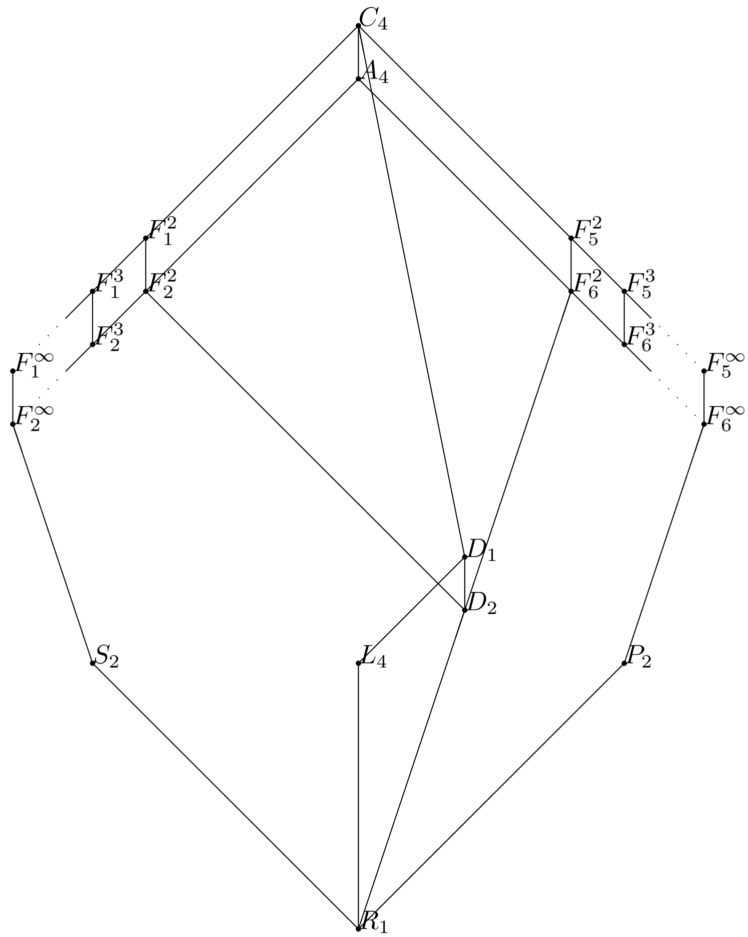


5.4. Monoidal interval $\mathrm{Int}(\{e, c_1\})$ or the $[\alpha, \beta]$-systems
It is exactly the dual of the interval $\mathrm{Int}(\{e, c_0\})$.

5.5. Monoidal interval $\mathrm{Int}(\{e, \neg\})$ or the $[\alpha, \delta]$-systems



5.6. Monoidal interval $\mathrm{Int}(\{e\})$ or the $[\alpha]$-systems.



6. Inclusion-exclusion tables

All the clones except for the ones of the form F_i^n can be distinguished from each other using only the operations c_0, c_1, $\neg$, $\wedge$, $\vee$, majority and ternary addition. Table 6 is the inclusion-exclusion diagram describing which clones contain which of the above operations except $\neg$. Those clones containing $\neg$ are underlined. For example $c_0, c_1, \neg, +^{(3)} \in L_1$ and $\vee, \wedge, maj \notin L_1$.

References

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	Generators	Relation	Monoid
C_1	$\langle \vee, \wedge, \neg \rangle$	$\emptyset$	$\{e, c_0, c_1, \neg\}$
$\hat{C}_2$	$\langle \wedge, \leftrightarrow \rangle$	$\{1\}$ [3]	$\{e, c_1\}$
$\hat{C}_3$	$\langle \vee, + \rangle$	$\{0\}$ [3]	$\{e, c_0\}$
C_4	$\langle \vee, \wedge, +^{(3)} \rangle$	$\{\binom{0}{1}\}$	$\{e\}$
$\hat{A}_1$	$\langle \vee, \wedge, c_0, c_1 \rangle$	$\leq$ [3]	$\{e, c_0, c_1\}$
A_2	$\langle \vee, \wedge, c_1 \rangle$	$\left\{ \begin{pmatrix} 0\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\1 \end{pmatrix} \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right\}$	$\{e, c_1\}$
A_3	$\langle \vee, \wedge, c_0 \rangle$	$\left\{ \begin{pmatrix} 0\\0\\0 \end{pmatrix} \begin{pmatrix} 0\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\1 \end{pmatrix} \right\}$	$\{e, c_0\}$
A_4	$\langle \vee, \wedge \rangle$	$\left\{ \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\1\\1 \end{pmatrix} \right\}$	$\{e\}$

TABLE 1. Description of the C and A clones

	Generators	Relation	Monoid
D_1	$\langle maj, +^{(3)} \rangle$	$\left\{ \begin{pmatrix} 0\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\0 \end{pmatrix} \right\}$	$\{e\}$
$\check{D}_2$	$\langle maj \rangle$	$\left\{ \begin{pmatrix} 0\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\0 \end{pmatrix} \begin{pmatrix} 1\\0\\1 \end{pmatrix} \right\}$	$\{e\}$
$\hat{D}_3$	$\langle maj, \neg \rangle$	$\neg^{\Box}$ [3]	$\{e, \neg\}$
$\hat{L}_1$	$\langle +, \neg \rangle$	$(+^{(3)})^{\Box}$ [3]	$\{e, c_0, c_1, \neg\}$
L_2	$\langle \leftrightarrow \rangle$	$\leftrightarrow^{\Box}$	$\{e, c_1\}$
L_3	$\langle + \rangle$	$+^{\Box}$	$\{e, c_0\}$
$\check{L}_4$	$\langle +^{(3)} \rangle$	$\left\{ \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\1\\0 \end{pmatrix} \begin{pmatrix} 1\\0\\1\\0 \end{pmatrix} \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix} \right\}$	$\{e\}$
L_5	$\langle +^{(3)}, \neg \rangle$		$\{e, \neg\}$

TABLE 2. Description of the D and L clones

	Generators	Relation	Monoid
F_1^2	$\langle maj, a \vee (b \wedge \neg c)\rangle$	$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$	$\{e\}$
F_1^n	$\langle h_n, a \vee (b \wedge \neg c)\rangle$		$\{e\}$
$\check{F}_1^\infty$	$\langle a \vee (b \wedge \neg c)\rangle$		$\{e\}$
F_2^2	$\langle maj, \vee\rangle$	$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$	$\{e\}$
$\check{F}_2^n$	$\langle h_n\rangle$		$\{e\}$
$\check{F}_2^\infty$	$\langle a \vee (b \wedge c)\rangle$		$\{e\}$
F_3^2	$\langle maj, c_1\rangle$	$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$	$\{e, c_1\}$
F_3^n	$\langle h_n, c_1\rangle$		$\{e, c_1\}$
F_3^∞	$\langle a \vee (b \wedge c), c_1\rangle$		$\{e, c_1\}$
$\hat{F}_4^2$	$\langle maj, a \vee \neg b\rangle$	$\{ \binom{0}{1} \binom{1}{0} \binom{1}{1} \}$ [3]	$\{e, c_1\}$
$\hat{F}_4^n$	$\langle h_n, a \vee \neg b\rangle$	$\{0,1\}^n \setminus \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$ [3]	$\{e, c_1\}$
F_4^∞	$\langle a \vee \neg b\rangle$		$\{e, c_1\}$
F_5^2	$\langle maj, a \wedge (b \vee \neg c)\rangle$	$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$	$\{e\}$
F_5^n	$\langle H_n, a \wedge (b \vee \neg c)\rangle$		$\{e\}$
$\check{F}_5^\infty$	$\langle a \wedge (b \vee \neg c)\rangle$		$\{e\}$
F_6^2	$\langle maj, \wedge\rangle$	$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$	$\{e\}$
$\check{F}_6^n$	$\langle H_n\rangle$		$\{e\}$
$\check{F}_6^\infty$	$\langle a \wedge (b \vee c)\rangle$		$\{e\}$
F_7^2	$\langle maj, c_0\rangle$		$\{e, c_0\}$
F_7^n	$\langle H_n, c_0\rangle$		$\{e, c_0\}$
F_7^∞	$\langle a \wedge (b \vee c), c_0\rangle$		$\{e, c_0\}$
$\hat{F}_8^2$	$\langle maj, a \wedge \neg b\rangle$	$\{ \binom{0}{0} \binom{0}{1} \binom{1}{0} \}$ [3]	$\{e, c_0\}$
$\hat{F}_8^n$	$\langle H_n, a \wedge \neg b\rangle$	$\{0,1\}^n \setminus \left\{ \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\}$ [3]	$\{e, c_0\}$
F_8^∞	$\langle a \wedge \neg b\rangle$		$\{e, c_0\}$

TABLE 3. Description of the F clones

	Generators	Relation	Monoid
$\check{S}_2$	$\langle \vee \rangle$		$\{e\}$
S_4	$\langle \vee, c_1 \rangle$	$\left\{ \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 0\\1\\1\\1 \end{pmatrix} \begin{pmatrix} 1\\0\\1\\1 \end{pmatrix} \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix} \right\}$	$\{e, c_1\}$
S_5	$\langle \vee, c_0 \rangle$	$\left\{ \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\1\\1 \end{pmatrix} \right\}$	$\{e, c_0\}$
$\hat{S}_6$	$\langle \vee, c_0, c_1 \rangle$	$\left\{ \begin{pmatrix} 0\\0\\0 \end{pmatrix} \begin{pmatrix} 0\\1\\1 \end{pmatrix} \begin{pmatrix} 1\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\1 \end{pmatrix} \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right\}$ [3]	$\{e, c_0, c_1\}$
$\check{P}_2$	$\langle \wedge \rangle$		$\{e\}$
P_4	$\langle \wedge, c_0 \rangle$	$\left\{ \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix} \begin{pmatrix} 0\\1\\1\\1 \end{pmatrix} \right\}$	$\{e, c_0\}$
P_5	$\langle \wedge, c_1 \rangle$	$\left\{ \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix} \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix} \right\}$	$\{e, c_1\}$
$\hat{P}_6$	$\langle \wedge, c_0, c_1 \rangle$	$\left\{ \begin{pmatrix} 0\\0\\0 \end{pmatrix} \begin{pmatrix} 0\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\0 \end{pmatrix} \begin{pmatrix} 1\\0\\0 \end{pmatrix} \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right\}$ [3]	$\{e, c_0, c_1\}$

TABLE 4. Description of the S and P clones

	Generators	Relation	Monoid
R_1	$\langle e \rangle$	$\left\{ \begin{pmatrix} 0\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\0 \end{pmatrix} \begin{pmatrix} 1\\0\\0 \end{pmatrix} \right\}$ [2]	$\{e\}$
$\check{R}_4$	$\langle \neg \rangle$	$\left\{ \begin{pmatrix} 0\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\0 \end{pmatrix} \begin{pmatrix} 0\\1\\1 \end{pmatrix} \begin{pmatrix} 1\\0\\0 \end{pmatrix} \begin{pmatrix} 1\\0\\1 \end{pmatrix} \begin{pmatrix} 1\\1\\0 \end{pmatrix} \right\}$	$\{e, \neg\}$
$\check{R}_6$	$\langle c_1 \rangle$	$\left\{ \begin{pmatrix} 0\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\0 \end{pmatrix} \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right\}$	$\{e, c_1\}$
$\check{R}_8$	$\langle c_0 \rangle$	$\left\{ \begin{pmatrix} 0\\0\\0 \end{pmatrix} \begin{pmatrix} 0\\1\\1 \end{pmatrix} \begin{pmatrix} 1\\0\\1 \end{pmatrix} \right\}$	$\{e, c_0\}$
R_{11}	$\langle c_0, c_1 \rangle$	$\left\{ \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix} \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix} \right\}$	$\{e, c_0, c_1\}$
$\hat{R_{13}}$	$\langle c_0, c_1, \neg \rangle$	$\left\{ \begin{pmatrix} 0\\0\\0 \end{pmatrix} \begin{pmatrix} 0\\0\\1 \end{pmatrix} \begin{pmatrix} 0\\1\\0 \end{pmatrix} \begin{pmatrix} 1\\0\\1 \end{pmatrix} \begin{pmatrix} 1\\1\\0 \end{pmatrix} \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right\}$	$\{e, c_0, c_1, \neg\}$

TABLE 5. Description of the R clones

			c_0				$\overline{c_0}$			
			$\wedge$		$\overline{\wedge}$		$\wedge$		$\overline{\wedge}$	
			maj	$\overline{maj}$	maj	$\overline{maj}$	maj	$\overline{maj}$	maj	$\overline{maj}$
c_1	$\vee$	$+^{(3)}$	$\underline{C_1}$				C_2		F_4^2	$F_4^n F_4^\infty$
		$\overline{+^{(3)}}$	A_1			S_6	A_2		F_3^2	$F_3^n F_3^\infty$ S_4
	$\overline{\vee}$	$+^{(3)}$				$\underline{L_1}$				L_2
		$\overline{+^{(3)}}$		P_6		$\underline{R_{13}}$ R_{11}		P_5		R_6
$\overline{c_1}$	$\vee$	$+^{(3)}$	C_3				C_4		F_1^2	$F_1^n F_1^\infty$
		$\overline{+^{(3)}}$	A_3			S_5	A_4		F_2^2	$F_2^n F_2^\infty$ S_2
	$\overline{\vee}$	$+^{(3)}$				L_3			$\underline{D_3}$ D_1	$\underline{L_5}$ L_4
		$\overline{+^{(3)}}$	F_8^2 F_7^2	$F_8^n F_8^\infty$ $F_7^n F_7^\infty$ P_4		R_8	F_5^2 F_6^2	$F_5^n F_5^\infty$ $F_6^n F_6^\infty$ P_2	D_2	$\underline{R_4}$ R_1

To distinguish between the clones grouped together in the table, we use the following operations:

- $a \vee (b \wedge c) \in F_2^\infty, F_3^\infty$, but $a \vee (b \wedge c) \notin S_2, S_4$
- $a \vee (b \wedge \neg c) \in F_1^n, F_1^\infty$, but $a \vee (b \wedge \neg c) \notin F_2^n, F_2^\infty$ for all $n \geq 2$
- $a \vee \neg b \in F_4^n, F_4^\infty$, but $a \vee \neg b \notin F_3^n, F_3^\infty$ for all $n \geq 2$
- $h_n \in F_i^n$, but $h_n \notin F_i^{n+1}, F_i^\infty$ for $i \in \{1, 2, 3, 4\}$ and for all $n \geq 2$

and dually:

- $a \wedge (b \vee c) \in F_6^\infty, F_7^\infty$, but $a \wedge (b \vee c) \notin P_2, P_4$
- $a \wedge (b \vee \neg c) \in F_5^n, F_5^\infty$, but $a \wedge (b \vee \neg c) \notin F_6^n, F_6^\infty$ for all $n \geq 2$
- $a \wedge \neg b \in F_8^n, F_8^\infty$, but $a \wedge \neg b \notin F_7^n, F_7^\infty$ for all $n \geq 2$
- $H_n \in F_i^n$, but $H_n \notin F_i^{n+1}, F_i^\infty$ for $i \in \{5, 6, 7, 8\}$ and for all $n \geq 2$

TABLE 6. Inclusion - exclusion table