Exact Discretization of the Solution to the Geometric Brownian Motion Stochastic Differential Equation

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Let P_t represent the time series of market prices of the underlying, μ be its mean continuous log-return, σ be its instantaneous volatility and W_t be a Wiener process.

Here is the stochastic differential equation for the geometric Brownian motion:

$$\frac{dP_t}{P_t} = \mu dt + \sigma dW_t$$

Here is the exact solution to the equation:

$$\ln\left(\frac{P_t}{P_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

The discretization of this solution over a small but finite interval δ is given by the following:

$$\ln\left(\frac{P_{t+\delta}}{P_t}\right) = \left(\mu - \frac{\sigma^2}{2}\right) \delta + \sigma W_t$$

where W_t amounts to a standard normal variate Z_t times the square root of the time interval δ, so that $W_t = Z_t\sqrt{\delta}$.

If the time to maturity is T, the number of time steps corresponding to the time interval δ is given by $n = {}^{T}\!/\!{}_{\delta}$. Thus,

$$\ln\left(\frac{P_T}{P_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{\delta}\sum_{k=1}^{n} Z_k$$

When one perform simulations, a path is represented by the foregoing formula, but there are as many instances of this formula as there are paths to simulate, so that even if the deterministic part of the formula is the same from path to path, the stochastic part of the formula, the Z_k, have to be generated anew for each instance of the formula, so that there are n times the number of simulations standard normal random numbers to generate in order to generate one path per simulation. Of course, the bigger the number of simulated paths and the smaller the δ, the more realistic are the results of the simulation.