From “How should I structure my code?” to Euler’s Pentagonal Numbers: A Problem in Resequencing Buffer Sizing

Peter Rabinovitch
Abstract

• In this work in progress we discuss a real-world problem about resequencing buffer sizing in network processor based router design, which when abstracted leads to some interesting combinatorics.
Who am I?

- PhD student (probability) @ Carleton
- Senior Research Scientist @ Alcatel-Lucent Bell Labs

Really interesting problems

Engineering

Mathematics
What is Resequencing?

- Origin of the Problem
- Other Approaches
- *Weighted Random Permutation Approach*
Your computer doesn’t care…

• Up to 5 or 10%, no real effect on end systems

But Light Reading does…

• Juniper test results (2001)
Intel Castine (IXP 2800)

- 16 cores, 8 threads each
- “By 2010 100s of cores, dozens of threads”
- Put a resequencing buffer on egress, but how big?
Existing resequencing literature

• Baccelli et al. (1984) “An End-to-End Approach to the Resequencing Problem”
• David Tse & Ye Xia (2004) “On the Large Deviation of Resequencing Queue Size: 2-M/M/1 Case”
• Jun Li (2006) “Mean Value Analysis of Resequencing Delay in a Discrete-time Queueing Model”
Standard approach

- M/M/Infinity or Geo/Geo/Infinity
  - Determine $P[\text{packet i leaves before packet j}]$
Different (combinatorics) approach

- Strip away *time*
- Inspired by work of O’Connell (“Queues, Stores, and Tableaux”) and LIS research (Tracy-Widom, etc), Baryshnikov (“GUEs & Queues”)
The Model - a few definitions

- *Permutation* $\pi$ of $[n]$ is an ordering of $\{1,2,\ldots,n\}$
- The pair $\{i,j\}$ is an *inversion* of $\pi=(p_1,p_2,p_3,\ldots,p_n)$ if $i<j$ but $p_i>p_j$
- The number of inversions $i(\pi)$
- The $i^{th}$ element of the *inversion table* of $\pi$ is the number of elements greater than $i$ to the left of $i$.
- There is a bijection between permutations & inversion tables
The Model - Example for the definitions

- Let $n=4$
- $\pi=(1,3,4,2)$
- Inversions are $\{2,4\}, \{3,4\}$
- $i(\pi)=2$
- Inversion table of $\pi$ is $(0,0,0,2)$

Note 1$^\text{st}$ element is always 0.
The Model – key result

• The largest element of the inversion table is the number of buffer spaces needed in a resequencing buffer

• Example:
  – \( \pi=(1,3,4,2) \)
  – 1 arrives, then it can go. 3 arrives, it must wait. 4 arrives it must wait. 2 arrives it can go. Then 3 can go. Then 4 can go.
  – Maximum element of the inversion table is 2.

• We denote it by \( M(\pi) \)
What is the probability of a permutation?

• If we choose uniform, then on average, the maximum element of the inversion table is $n - \sqrt{n\pi/2}$ – not realistic

• More appropriate
  – packets are more likely to swap places with packets that are close by, than those that are far away
  – A permutation with fewer swaps is more likely than one with many swaps
The Model – Permutation probability

• Previous assumptions agree with "Measuring Packet Reordering" by Bellardo and Savage, suggest \( p < 10\% \)

• \( P[I=i] \sim p^i \)
  – where \( 0 < p < 1 \)

• Our target is \( E[M] \)
Let us calculate...

\[ P[M(\pi) = m] = \sum_i P[M(\pi) = m \mid I(\pi) = i]P[I(\pi) = i] \]

- \( P[I(\pi) = i] \) is easy

\[ P[I(\pi) = i] = \frac{1}{Z} \frac{1}{b(n, i)} p^i \]

- \( Z \) is a normalizing constant
- \( b(n, i) \) is the number of permutations on [n] with exactly i inversions.
  - \( b(n, i) \) involves Euler’s Pentagonal numbers: \( j(3j \pm 1)/2 \)
So the question becomes...

• List the possible maximum elements of an inversion table where the number of inversions is i.

• Thus we need to find the number of integer solutions $s(n,m,i)$ of the following system of equations, for each $m=0,1,...,n-1$

$$x_1 + x_2 + \cdots + x_{n-1} = i$$

$$\forall j: 0 \leq x_j \leq \min(j, m)$$

$$\exists j: x_j = m$$
A Picture is Worth…

- Put $i$ balls into this shape, bottom justified, where at least one column hits the upper boundary
A Picture is Worth…

- Put t of them into the green part, i-t into the blue
A Picture is Worth…

- Put t of them into the green part, i-t into the blue

\[ b(m, i - t) \]

\[ a(n - m, t, m) \]
a(n,i,m)

• $a(n,i,m) = \# \text{ of solutions of }$

\begin{align*}
x_1 + x_2 + \cdots + x_n &= i
\end{align*}

where

\begin{align*}
\forall j : x_j &\leq m
\end{align*}

and

\begin{align*}
\exists j : x_j &= m
\end{align*}

i.e. a weak composition of $i$ into $n$ parts with at least one part equaling $m$, and all parts less than or equal to $m$. 
Putting it all together...(1)

\[ E[M] = \sum_{m=1}^{n-1} m P[M = m] \]

\[ P[M = m] = \frac{1}{Z} \sum_{i=0}^{2} s(n,m,i) \frac{1}{b(n,i)} p^i \]

\[ Z = \frac{1 - p^2}{1 - p} \]

\[ b(n,i) = [x^i] \prod_{k=1}^{n-1} \sum_{j=0}^{k} x^j \]
Putting it all together…(2)

\[ s(n, m, i) = \sum_{t=\text{Max}\left\{i-\frac{m(m-1)}{2}, m\right\}}^{\text{Min}\{(n-m)m,i\}} b(m, i-t) a(n-m, t, m) \]

\[ a(n, i, m) = c(n, i, m) - c(n, i, m-1) \]

\[ c(n, i, m) = \sum_{j=0}^{\text{Min}\left\{n, \frac{i}{m+1}, \frac{n+i-1}{m+1}\right\}} (-1)^j \binom{n}{j} \binom{n+i-j(m+1)-1}{n-1} \]
Putting it all together... in summary

\[ E[M] = \frac{1}{Z} \sum_{m} \sum_{t} \sum_{m} a(n - m, t, m) \frac{b(m, i - t)}{b(n, i)} m p^i \]
n=4

Expected resequencing buffer required

uniform

model
n=15

Expected resequencing buffer required

uniform

model

p
Preliminary result

Expected resequencing buffer required

\[ n \]
Next Steps

• Bounds/asymptotics in order to understand the expected value, distribution
• Comparison to ‘standard’ models
  – i.e. the M/M/Infinity, Geo/Geo/Infinity models
• Other models for reordering
  – Markov dependence
  – $P[1 \text{ large inversion}] < P[\text{many small inversions}]$
Thanks!